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APPROXIMATIONS TO EULER'S CONSTANT

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Abstract

We study a problem of finding good approximations to Euler's constant $\gamma = \lim_{n\to\infty} S_n$, where $S_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1)$, by linear forms in logarithms and harmonic numbers. In 1995, C. Elsner showed that slow convergence of the sequence S_n can be significantly improved if S_n is replaced by linear combinations of S_n with integer coefficients. In this paper, considering more general linear transformations of the sequence S_n we establish new accelerating convergence formulae for γ. Our estimates sharpen and generalize recent Elsner's, Rivoal's and author's results.

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1. Introduction

Let $\alpha \geq 0$ be a real number and

$$\gamma_\alpha = \sum_{k=1}^{\infty} \left(\frac{1}{k+\alpha} - \log\left(\frac{k+\alpha+1}{k+\alpha} \right) \right).$$

We denote the partial sum of the above series by

$$\begin{aligned} S_n(\alpha) &= \sum_{k=1}^{n} \left(\frac{1}{k+\alpha} - \log\left(\frac{k+\alpha+1}{k+\alpha} \right) \right) \\ &= \sum_{k=1}^{n} \frac{1}{k+\alpha} - \log(\alpha+n+1) + \log(\alpha+1) \end{aligned} \tag{1}$$

and $S_n := S_n(0)$. It easily follows (see [12, formula (2)]) that

$$\lim_{n\to\infty} S_n(\alpha) = -\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} + \log(\alpha+1) = -\psi(\alpha+1) + \log(\alpha+1),$$

where $\psi(\alpha)$ is the logarithmic derivative of the gamma function (or the digamma function) and therefore,

$$\gamma_\alpha = \log(\alpha+1) - \psi(\alpha+1).$$

In particular, $\gamma_0 = -\psi(1) = \gamma = 0.577215\ldots$, where γ is Euler's constant. It is well-known that the sequence S_n slowly converges to the Euler constant γ (see, for details, [7])

$$\gamma = S_n + O(n^{-1}).$$

In 1995, Elsner [1] found out that γ can be approximated by linear combinations of partial sums (1) with integer coefficients

$$\left| \gamma - \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{k+n+\tau-1}{k+\tau-1} S_{k+\tau-1} \right| \leq \frac{1}{2n\tau\binom{n+\tau}{n}}, \qquad \tau, n \in \mathbb{N} \tag{2}$$

and this inequality exhibits geometric convergence if $\tau = O(n)$. Formulas (2) for $\tau > n$ were generalized by Rivoal in [10], where, in particular, it was shown that

$$\left| \gamma - \frac{1}{2^n} \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} \binom{2k+2n}{n} S_{2k+n} \right| = O\left(\frac{1}{n 27^{n/2}} \right), \quad n \to \infty.$$

Another such kind of formula

$$\gamma - \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} \binom{n+k}{k} S_{k+n} = \frac{1}{4^{n+o(n)}}, \quad n \to \infty$$

was proved in [6]. Recently, C. Elsner [2] presented a two-parametric series transformation of the sequence S_n

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+\tau_1+k}{n} S_{k+\tau_2-1} \tag{3}$$

converging more rapidly to γ, when $\tau_2 > \tau_1 + 1$ and n increases, than in the case $\tau_2 = \tau_1 + 1$ considered in (2).

In this paper, we consider a more general series transformation of the type

$$\frac{n_1!\dots n_m!}{N!\,r^N}\sum_{k=0}^{N}(-1)^{N+k}\binom{N}{k}\binom{rk+n_1+\tau_1}{n_1}\dots\binom{rk+n_m+\tau_m}{n_m}S_{rk+\tau_0} \tag{4}$$

with $n_1,\dots,n_m\in\mathbb{N}$, $\tau_0,\tau_1,\dots,\tau_m\in\mathbb{N}_0$, and $N=\sum_{j=1}^m n_j$, and give new accelerating convergence formulae for Euler's constant γ. In particular, we show (see Theorem 2 and Corollary 1 below) that if τ_1,τ_2 are linear functions of n, then the sum (3) converges to γ at the least geometric rate and represents the best approximation in the set of all the sums (3) with a fixed value of $\lim\limits_{n\to\infty}\tau_2/n$, provided that $\lim\limits_{n\to\infty}2(\tau_2-\tau_1)/n=1$.

2. Statement of the main results

As usual, we denote the Gauss hypergeometric function (see, for details, [9]) by

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right)=\sum_{\nu=0}^{\infty}\frac{(a)_\nu(b)_\nu}{\nu!\,(c)_\nu}z^\nu,$$

where $(\lambda)_\nu$ is the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1, & \nu=0;\\ \lambda(\lambda+1)\dots(\lambda+\nu-1), & \nu\in\mathbb{N}.\end{cases}$$

We then prove the following theorems:

Theorem 1. *Let $n_1,\dots,n_m\in\mathbb{N}$, $\tau_0,\tau_1,\dots,\tau_m\in\mathbb{N}_0$, $0\le\tau_0-\tau_m\le n_m$, $n_m+\tau_m\ge n_j+\tau_j$, $j=1,\dots,m-1$, and $N=\sum_{j=1}^m n_j$. Then*

$$\begin{aligned}&\left|\frac{N!\,(-r)^N}{n_1!\dots n_m!}\gamma-\sum_{k=0}^{N}(-1)^k\binom{N}{k}\binom{rk+n_1+\tau_1}{n_1}\dots\binom{rk+n_m+\tau_m}{n_m}S_{rk+\tau_0}\right|\\ &=\prod_{j=1}^{m}\binom{n_m+\tau_m-\tau_j}{n_j}\int_0^1\!\!\int_0^1\frac{x^{n_m+\tau_m}(1-x^r)^N t^{n_m+\tau_m-\tau_0}(1-t)^{\tau_0-\tau_m}\omega(t)}{(1-t+xt)^{n_m+1}}\\ &\times\left|Q_m\left(\frac{xt}{1-t+xt}\right)\right|\,dxdt,\end{aligned}\tag{5}$$

where

$$\omega(t)=\frac{1}{t(\log^2(1/t-1)+\pi^2)} \tag{6}$$

and $Q_m(y)$ is a polynomial of degree $N-n_m$ given by the formula

$$Q_m(y)=\sum_{k_1=0}^{n_1}\dots\sum_{k_{m-1}=0}^{n_{m-1}}\prod_{j=1}^{m-1}\frac{(-n_j)_{k_j}(1+n_m+\tau_m-\tau_{j+1})_{k_1+\ldots+k_j}}{k_j!\,(1+n_m+\tau_m-n_j-\tau_j)_{k_1+\ldots+k_j}}y^{k_j} \tag{7}$$

if $m\ge 2$, and $Q_1(y)\equiv 1$.

Theorem 2. *Let $b,c,r\in\mathbb{N}$, $a\in\mathbb{N}_0$, $0\le b-a\le c$. Then for $n\in\mathbb{N}$ we have*

$$\left|\gamma-\frac{1}{r^{cn}}\sum_{k=0}^{cn}(-1)^{k+cn}\binom{cn}{k}\binom{rk+(a+c)n}{cn}S_{rk+bn}\right|<\left(\frac{b^{\frac{b}{r}}(c+a-b)^{c+a-b}(b-a)^{b-a}}{(b+cr)^{c+\frac{b}{r}}}\right)^n \tag{8}$$

(Here and throughout the paper 0^0 is treated as 1.)

If b, c, r are fixed, then the minimum of the right-hand side of (8) is attained when $b-a=c/2$ and in this case we have

Corollary 1. *Let $b, c, r, n \in \mathbb{N}$ and $b \geq c$. Then*

$$\left|\gamma - \frac{1}{r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk+(b+c)n}{2cn} S_{rk+bn}\right| < \left(\frac{b^{\frac{b}{r}} c^{2c}}{(b+2cr)^{2c+\frac{b}{r}}}\right)^n.$$

Theorem 3. *Let $b, c, r \in \mathbb{N}$, $a \in \mathbb{N}_0$ and $0 \leq b-a \leq c$. Then for any positive integer $n \geq 2/c$ one has*

$$\left|\gamma - \frac{((cn)!)^2}{(2cn)!\, r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk+(a+c)n}{cn}^2 S_{rk+bn}\right|$$
$$< cn \left(\frac{b^{\frac{b}{r}} c^c (c+a-b)^{c+a-b} (b-a)^{b-a}}{(b+2cr)^{2c+\frac{b}{r}}}\right)^n.$$

By the similar argument as above putting $a=b-c/2$ we get a sharper bound than in Corollary 1.

Corollary 2. *Let $b, c, r, n \in \mathbb{N}$, $2b \geq c$, and c is even. Then*

$$\left|\gamma - \frac{((cn)!)^2}{(2cn)!\, r^{2cn}} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk+\left(b+\frac{c}{2}\right)n}{cn}^2 S_{rk+bn}\right| < cn \left(\frac{b^{\frac{b}{r}} c^{2c}}{2^c (b+2cr)^{2c+\frac{b}{r}}}\right)^n.$$

For example, setting $b=c=4, r=1$ we get the following estimate:

Corollary 3. *For any positive integer n one has*

$$\left|\gamma - \frac{(4n)!^2}{(8n)!} \sum_{k=0}^{8n} (-1)^k \binom{8n}{k} \binom{k+6n}{4n}^2 S_{k+4n}\right| < \frac{4n}{(2^4\ 3^{12})^n} < 4n(0.00000012)^n.$$

Theorem 4. *Let $n_1, \ldots, n_m \in \mathbb{N}$, $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{N}_0$, $0 \leq \tau_0 - \tau_m \leq n_m$, $n_m + \tau_m \geq \tau_{j+1} > n_j + \tau_j$, $j=1, \ldots, m-1$, and $N = \sum_{j=1}^m n_j$, Then*

$$\left|\frac{N!\,(-r)^N}{n_1! \ldots n_m!} \gamma - \sum_{k=0}^{N} (-1)^k \binom{N}{k} \binom{rk+n_1+\tau_1}{n_1} \cdots \binom{rk+n_m+\tau_m}{n_m} S_{rk+\tau_0}\right|$$
$$\leq \prod_{j=1}^{m} \binom{n_m+\tau_m-\tau_j}{n_j} \int_0^1\!\!\int_0^1 \frac{x^{n_m+\tau_m}(1-x^r)^N t^{n_m+\tau_m-\tau_0}(1-t)^{\tau_0-\tau_m} \omega(t)}{(1-t+xt)^{n_m+1}}\, dx dt.$$

Setting $\tau_{j+1} = n_j + \tau_j + 1$, $j=1, \ldots, m-1$, in Theorem 4 we get

Corollary 4. *Let $n_1,\dots,n_m\in\mathbb{N}$, $\tau_0,\tau_1\in\mathbb{N}_0$, $N=\sum_{j=1}^m n_j$, and $N-n_m+\tau_1+(m-1)\le\tau_0\le N+\tau_1+(m-1)$. Then*

$$\left|\gamma-\frac{n_1!\dots n_m!}{N!(-r)^N}\sum_{k=0}^N(-1)^k\binom{N}{k}\prod_{j=1}^m\binom{rk+n_1+\dots+n_j+\tau_1+j-1}{n_j}S_{rk+\tau_0}\right|$$
$$\le\prod_{j=1}^{m-1}\frac{N+j}{n_{j+1}+\dots+n_m+m-j}\times$$
$$\int_0^1\int_0^1\frac{x^{N+\tau_1+m-1}(1-x^r)^N t^{N+\tau_1+m-1-\tau_0}(1-t)^{\tau_0+n_m-N-\tau_1-m+1}\omega(t)}{r^N(1-t+xt)^{n_m+1}}\,dxdt$$

Theorem 5. *Let $m,c_1,\dots,c_m,r,b,n\in\mathbb{N}$, $a\in\mathbb{N}_0$, $C=\sum_{j=1}^m c_j$, and $a-c_m\le b-c\le a$. Then*

$$\left|\gamma-\frac{(c_1n)!\dots(c_mn)!}{(Cn)!(-r)^{Cn}}\sum_{k=0}^N(-1)^k\binom{Cn}{k}\prod_{j=1}^m\binom{rk+(a+c_1+\dots+c_j)n+j-1}{c_jn}\right.$$
$$\left.\times S_{rk+bn+m}\right|<M(\overline{c})\left(\frac{b^{\frac{b}{r}}C^C(C+a-b)^{C+a-b}(c_m+b-a-C)^{c_m+b-a-C}}{c_m^{c_m}(b+Cr)^{C+\frac{b}{r}}}\right)^n,$$

where $M(\overline{c})<C^{m-1}$ is some constant depending only on $c_1,\dots,c_m$.

Consider several illustrative examples of Theorem 5. Taking $c_1=\dots=c_m=2c$, $C=2mc$, $b=2mc$, $a=c$, $c\in\mathbb{N}$, we get

Corollary 5. *Let $c,m,r\in\mathbb{N}$. Then for any positive integer n one has*

$$\left|\gamma-\frac{(2cn)!^m}{(2mcn)!r^{2mcn}}\sum_{k=0}^{2mcn}(-1)^k\binom{2mcn}{k}\binom{rk+3cn+1}{2cn}\binom{rk+5cn+2}{2cn}\dots\right.$$
$$\left.\times\binom{rk+(2m+1)cn+m}{2cn}S_{rk+2mcn+m}\right|<\frac{m^m}{(m-1)!}\left(\frac{1}{4^c(r+1)^{2mc+\frac{2mc}{r}}}\right)^n$$

Setting $c_1=\dots=c_m=2c$, $C=2mc$, $b=(2m-1)c$, $a=2c$, $c\in\mathbb{N}$, we get

Corollary 6. *Let $c,m,r\in\mathbb{N}$. Then for any positive integer n one has*

$$\left|\gamma-\frac{(2cn)!^m}{(2mcn)!r^{2mcn}}\sum_{k=0}^{2mcn}(-1)^k\binom{2mcn}{k}\prod_{j=1}^m\binom{rk+2jcn+j}{2cn}S_{rk+(2m-1)cn+m}\right|$$
$$<\frac{m^m}{(m-1)!}\left(\frac{4^{-c}\left(1-\frac{1}{2m}\right)^{\frac{(2m-1)c}{r}}}{\left(r+1-\frac{1}{2m}\right)^{2mc+\frac{(2m-1)c}{r}}}\right)^n.$$

3. Analytical construction

We define the generalized Legendre polynomial by $A(x)=\sum_{k=0}^N A_kx^{rk}$ with

$$A_k=(-1)^{k+N}\binom{N}{k}\binom{rk+n_1+\tau_1}{n_1}\dots\binom{rk+n_m+\tau_m}{n_m}.$$

Lemma 1. *There holds*

$$A(1)=\sum_{k=0}^{N}A_k=\frac{N!\,r^N}{n_1!\dots n_m!}.$$

Proof. For the proof, let

$$R(t)=\frac{N!}{n_1!\dots n_m!}\frac{(rt-n_1-\tau_1)_{n_1}(rt-n_2-\tau_2)_{n_2}\dots(rt-n_m-\tau_m)_{n_m}}{t(t+1)\dots(t+N)}.$$

Such rational functions were considered earlier by the authors [4], [5] to derive explicit Padé approximations of the first and second kinds for polylogarithmic functions. As it is easily seen the rational function $R(t)$ has the following partial-fraction expansion:

$$R(t)=\sum_{k=0}^{N}\frac{A_k}{t+k},$$

from which it follows that

$$\sum_{k=0}^{N}A_k=\sum_{k=0}^{N}\operatorname*{res}_{t=-k}R(t)=-\operatorname*{res}_{t=\infty}R(t)=\frac{N!\,r^N}{n_1!\dots n_m!}. \qquad \square$$

Put

$$I(\alpha):=\int_0^1 x^{\tau_0+\alpha}A(x)\left(\frac{1}{1-x}+\frac{1}{\log x}\right)dx$$

Lemma 2. *There holds the equality*

$$I(\alpha)=\frac{N!\,r^N}{n_1!\dots n_m!}\,\gamma_\alpha-\sum_{k=0}^{N}A_kS_{rk+\tau_0}(\alpha).$$

Proof. Substituting

$$\frac{1}{1-x}+\frac{1}{\log x}=\int_0^1\frac{1-x^t}{1-x}\,dt,$$

we get

$$I(\alpha)=\int_0^1\int_0^1 x^{\tau_0+\alpha}A(x)\frac{1-x^t}{1-x}\,dtdx=\sum_{k=0}^{N}A_k\int_0^1\int_0^1\frac{x^{rk+\tau_0+\alpha}(1-x^t)}{1-x}\,dxdt.$$

Expanding $(1-x)^{-1}$ in a geometric series and applying Lemma 1 we find

$$\begin{aligned}
I(\alpha)&=\sum_{k=0}^{N}A_k\sum_{l=0}^{\infty}\int_0^1\int_0^1 x^{rk+\tau_0+l+\alpha}(1-x^t)\,dxdt\\
&=\sum_{k=0}^{N}A_k\sum_{l=0}^{\infty}\int_0^1\left(\frac{1}{rk+\tau_0+l+\alpha+1}-\frac{1}{rk+\tau_0+t+l+\alpha+1}\right)dt\\
&=\sum_{k=0}^{N}A_k\sum_{l=1}^{\infty}\left(\frac{1}{rk+\tau_0+l+\alpha}-\log\left(\frac{rk+\tau_0+l+\alpha+1}{rk+\tau_0+l+\alpha}\right)\right)\\
&=\sum_{k=0}^{N}A_k(\gamma_\alpha-S_{rk+\tau_0}(\alpha))=\frac{N!\,r^N}{n_1!\dots n_m!}\,\gamma_\alpha-\sum_{k=0}^{N}A_kS_{rk+\tau_0}(\alpha). \qquad \square
\end{aligned}$$

Next, we consider two differential operators

$$\begin{aligned}
S_{\tau,n}(f(x))&=\frac{(-1)^n}{n!}x^{-\tau}\left(x^{n+\tau}f(x)\right)^{(n)},\\
T_{\tau,n}(f(x))&=\frac{1}{n!}x^{n+\tau}\left(x^{-\tau}f(x)\right)^{(n)},
\end{aligned}$$

where τ is a real number and n is a non-negative integer. We show that $S_{\tau,n}$ and $T_{\tau,n}$ are adjoint operators in some sense.

Lemma 3. *Suppose that $f(x)$ is a polynomial vanishing at $x = 1$ with order at least n and $g(x) \in C^{\infty}(0,1) \cap L^1(0,1)$ satisfies the following boundary conditions:*

$$\lim_{x\to 0+} x^l g^{(l-1)}(x) = \lim_{x\to 1-} (1-x)^l g^{(l-1)}(x) = 0$$

for all $1 \le l \le n$. Then we have

$$\int_0^1 S_{\tau,n}(f(x)) \cdot g(x)\, dx = \int_0^1 f(x) \cdot T_{\tau,n}(g(x))\, dx.$$

Proof. The proof is analogous to the proof of Lemma 3.1 [3]. □

Lemma 4. *There holds*

$$I(\alpha) = \int_0^1\int_0^1 (1-x^r)^N \omega(t) T_{\tau_{m-1},n_{m-1}} \circ \ldots \circ T_{\tau_1,n_1} \circ T_{\tau_m,n_m} \left(\frac{x^{\tau_0+\alpha}}{1-(1-x)t} \right) dxdt$$

with the weight function $\omega(t)$ defined in (6).

Proof. Applying the following representation introduced by Prévost [8]:

$$\frac{1}{1-x} + \frac{1}{\log x} = \int_0^1 \frac{\omega(t)}{1-(1-x)t}\, dt,$$

we have

$$I(\alpha) = \int_0^1 \int_0^1 \frac{x^{\tau_0+\alpha}\omega(t)}{1-(1-x)t} A(x)\, dtdx.$$

As it easily follows the polynomial $A(x)$ can be written in the form

$$A(x) = S_{\tau_1,n_1} \circ S_{\tau_2,n_2} \circ \ldots \circ S_{\tau_m,n_m} \left((1-x^r)^N \right).$$

Since $A(x)$ is symmetric in pairs (τ_j, n_j) and does not depend on the order of differential operators S_{τ_j,n_j}, it is convenient for the sequel to be written as

$$A(x) = S_{\tau_m,n_m} \circ S_{\tau_1,n_1} \circ \ldots \circ S_{\tau_{m-1},n_{m-1}} \left((1-x^r)^N \right).$$

Now by Fubini's theorem and Lemma 3, we get the desired equality. □

We also need the following simple lemma, which will be used for estimation purposes.

Lemma 5. *Let $a, b, c, d, r, s \in \mathbb{R}$, $r, s, d > 0$, and $b + d \ge a + c \ge b \ge 0$. Then the function*

$$f(x,t) = \frac{x^{a+c}(1-x^r)^{sc} t^{c+a-b}(1-t)^{b+d-c-a}}{(1-t+xt)^d}$$

attains its maximum in $[0,1] \times [0,1]$ at the unique point

$$x_0 = \left(\frac{b}{b+scr} \right)^{\frac{1}{r}}, \qquad t_0 = \frac{c+a-b}{c+a-b+x_0(b+d-a-c)}$$

and

$$\max_{0\le x,t\le 1} f(x,t) = f(x_0,t_0) = \frac{b^{\frac{b}{r}}(scr)^{sc}(c+a-b)^{c+a-b}(b+d-a-c)^{b+d-a-c}}{d^d (b+scr)^{sc+\frac{b}{r}}}.$$

4. Proof of Theorem 1

Lemma 6. *Let $x,t\in(0,1)$, $\tau_0,n_m,\tau_m\in\mathbb{N}_0$, and $\tau_m\le\tau_0\le n_m+\tau_m$. Then*

$$T_{\tau_m,n_m}\left(\frac{x^{\tau_0}}{1-(1-x)t}\right)=(-1)^{n_m}\frac{x^{n_m+\tau_m}t^{n_m+\tau_m-\tau_0}(t-1)^{\tau_0-\tau_m}}{(1-(1-x)t)^{n_m+1}}.$$

Proof. Clearly,

$$T_{\tau_m,n_m}\left(\frac{x^{\tau_0}}{1-t+xt}\right)=\frac{x^{n_m+\tau_m}}{n_m!}\left(\frac{x^{\tau_0-\tau_m}}{1-t+xt}\right)^{(n_m)}.$$

Decomposing the fraction $\frac{x^{\tau_0-\tau_m}}{1-t+xt}$ into the sum

$$\frac{x^{\tau_0-\tau_m}}{1-t+xt}=p(x)+\left(\frac{t-1}{t}\right)^{\tau_0-\tau_m}\frac{1}{1-t+xt},$$

where $p(x)$ is a polynomial of degree not exceeding $\tau_0-\tau_m-1$, and differentiating it n_m times, we get the required statement. □

Lemma 7. *Under the hypothesis of Theorem 1 one has*

$$\begin{aligned}&T_{\tau_{m-1},n_{m-1}}\circ\ldots\circ T_{\tau_1,n_1}\circ T_{\tau_m,n_m}\left(\frac{x^{\tau_0}}{1-(1-x)t}\right)=(-1)^{n_m}\\ &\times\prod_{j=1}^{m}\binom{n_m+\tau_m-\tau_j}{n_j}\frac{x^{n_m+\tau_m}t^{n_m+\tau_m-\tau_0}(t-1)^{\tau_0-\tau_m}}{(1-t+xt)^{n_m+1}}Q_m\left(\frac{xt}{1-t+xt}\right),\end{aligned}\tag{9}$$

where the polynomial $Q_m(y)$ is defined in (7).

Proof. If $m=1$, then (9) easily follows by Lemma 6. Suppose $m\ge 2$. Then consecutive calculation of the n_jth derivatives with respect to x by Leibniz' rule for $j=1,2,\ldots,m-1$

$$\begin{aligned}\frac{x^{\tau_j+n_j}}{n_j!}\left(\frac{t^k x^{n_m+\tau_m+k-\tau_j}}{(1-t+xt)^{n_m+1+k}}\right)^{(n_j)}&=\binom{n_m+\tau_m-\tau_j}{n_j}\frac{x^{n_m+\tau_m}}{(1-t+xt)^{n_m+1}}\\ \times\sum_{k_j=0}^{n_j}\frac{(-n_j)_{k_j}(n_m+1)_{k+k_j}(1+n_m+\tau_m-\tau_j)_k}{k_j!\,(n_m+1)_k(1+n_m+\tau_m-\tau_j-n_j)_{k+k_j}}&\left(\frac{xt}{1-t+xt}\right)^{k+k_j}\end{aligned}$$

readily leads to the formula (9). □

Now Theorem 1 easily follows from Lemmas 4 and 7.

5. Proof of Theorem 2

If we put $m=1, n_1=cn, \tau_1=an, \tau_0=bn, n\in\mathbb{N}$, in Theorem 1, we get

$$\begin{aligned}&\left|\gamma-\frac{1}{r^{cn}}\sum_{k=0}^{cn}(-1)^{k+cn}\binom{cn}{k}\binom{rk+(a+c)n}{cn}S_{rk+bn}\right|\\ &\le\frac{1}{r^{cn}}\int_0^1\int_0^1\frac{(1-x^r)^{cn}x^{(a+c)n}t^{(c+a-b)n}(1-t)^{(b-a)n}\omega(t)}{(1-t+xt)^{cn+1}}\,dxdt\\ &\le\frac{1}{r^{cn}}\left(\max_{0\le x,t\le 1}f(x,t)\right)^n\int_0^1\int_0^1\frac{\omega(t)}{1-t+xt}dtdx=\frac{\gamma}{r^{cn}}\left(\max_{0\le x,t\le1}f(x,t)\right)^n\end{aligned}$$

with

$$f(x,t)=\frac{x^{a+c}(1-x^r)^c t^{c+a-b}(1-t)^{b-a}}{(1-t+xt)^c}.$$

Here we used the fact (see [8, formula 2.6]) that

$$\gamma = \int_0^1 \left(\frac{1}{\log x} + \frac{1}{1-x} \right) dx.$$

Now, since $\gamma < 1$, by Lemma 5 with $s = 1, d = c$, the theorem follows. □

6. Proofs of Theorems 3 and 4

To estimate the speed of convergence of quantities (4) to γ as $N \to \infty$ we need an upper bound for the polynomial $Q_m(y)$. In some situations it is possible to get suitable estimations.

First, we consider the case $m = 2$, $n_1 = n_2$, $\tau_1 = \tau_2$. Then by Theorem 1, we get

$$\begin{aligned} I := & \left| \frac{(2n_1)!\, r^{2n_1}}{(n_1!)^2} \gamma - \sum_{k=0}^{2n_1} (-1)^k \binom{2n_1}{k} \binom{rk + n_1 + \tau_1}{n_1}^2 S_{rk+\tau_0} \right| \\ = & \int_0^1 \int_0^1 \frac{x^{n_1+\tau_1}(1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0}(t-1)^{\tau_0-\tau_1}\omega(t)}{(1-t+xt)^{n_1+1}} \, |Q_2(y)| \, dxdt \end{aligned}$$

with $y = xt/(1-t+xt)$. The polynomial

$$Q_2(y) = {}_2F_1 \left(\left. \begin{matrix} -n_1, n_1+1 \\ 1 \end{matrix} \right| y \right) = \frac{1}{n_1!} \left(\frac{d}{dy} \right)^{n_1} \left(y^{n_1}(1-y)^{n_1} \right)$$

is a shifted Legendre polynomial $P_{n_1}(u)$ formally identified as follows:

$$Q_2(y) = P_{n_1}(1-2y).$$

By the well-known inequality (see [11, p.162])

$$|P_{n_1}(u)| \le 1, \qquad -1 \le u \le 1,$$

it follows that

$$I \le \int_0^1 \int_0^1 \frac{x^{n_1+\tau_1}(1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0}(1-t)^{\tau_0-\tau_1}\omega(t)}{(1-t+xt)^{n_1+1}} \, dxdt.$$

Now, setting $n_1 = cn$, $\tau_1 = an$, $\tau_0 = bn$ with $c, b \in \mathbb{N}$, $a \in \mathbb{N}_0$, and $0 \le b - a \le c$, we get

$$\left| \frac{(2cn)!\, r^{2cn}}{((cn)!)^2} \gamma - \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{rk + (a+c)n}{cn}^2 S_{rk+bn} \right| \le \gamma \left(\max_{0 \le x,t \le 1} f(x,t) \right)^n,$$

where

$$f(x,t) = \frac{x^{c+a}(1-x^r)^{2c} t^{a+c-b}(1-t)^{b-a}}{(1-t+xt)^c}.$$

By Lemma 5, the function $f(x,t)$ takes its maximum in $[0,1] \times [0,1]$ at the unique point (x_0, t_0), at which

$$f(x_0, t_0) = \frac{b^{\frac{b}{r}} (4cr^2)^c (c+a-b)^{c+a-b} (b-a)^{b-a}}{(b+2cr)^{2c+\frac{b}{r}}}.$$

Since for any positive integer $n \ge 2$

$$\gamma \frac{(n!)^2}{(2n)!} \le \frac{n}{4^n},$$

Theorem 3 follows. □

Another interesting case is described by the following lemma.

Lemma 8. *Let* $n_1,\dots,n_m \in \mathbb{N}$, $\tau_0,\tau_1,\dots,\tau_m \in \mathbb{N}_0$, *and* $n_m+\tau_m \ge \tau_{j+1} > n_j+\tau_j$, $j = 1,\dots,m-1$. *Then*

$$
\begin{aligned}
Q_m(y) &= \prod_{j=1}^{m-1} \frac{(n_m+\tau_m-n_j-\tau_j)!}{(n_m+\tau_m-\tau_{j+1})!(\tau_{j+1}-n_j-\tau_j-1)!}\times \\
&\int_0^1\!\!\dots\!\int_0^1 \prod_{j=1}^{m-1}(1-yu_j\dots u_{m-1})^{n_j}u_j^{n_m+\tau_m-\tau_{j+1}}(1-u_j)^{\tau_{j+1}-n_j-\tau_j-1}du_1\dots du_{m-1}.
\end{aligned}
\tag{10}
$$

Moreover, $0 \le Q_m(y) \le 1$ *for* $y \in [0,1]$.

Proof. Denoting the integral on the right-hand side of (10) by J and substituting

$$
\prod_{j=1}^{m-1}(1-yu_ju_{j+1}\dots u_{m-1})^{n_j} = \sum_{k_1=0}^{n_1}\dots\sum_{k_{m-1}=0}^{n_{m-1}}\prod_{j=1}^{m-1}\frac{(-n_j)_{k_j}y^{k_j}u_j^{k_1+\dots+k_j}}{k_j!},
$$

we get

$$
\begin{aligned}
J &= \sum_{k_1=0}^{n_1}\dots\sum_{k_{m-1}=0}^{n_{m-1}}\prod_{j=1}^{m-1}\frac{(-n_j)_{k_j}y^{k_j}}{k_j!}\int_0^1 u_j^{k_1+\dots+k_j+n_m+\tau_m-\tau_{j+1}}\times \\
&\times(1-u_j)^{\tau_{j+1}-n_j-\tau_j-1}\,du_j = \sum_{k_1=0}^{n_1}\dots\sum_{k_{m-1}=0}^{n_{m-1}}\prod_{j=1}^{m-1}\frac{(-n_j)_{k_j}y^{k_j}}{k_j!}\times \\
&\times\frac{\Gamma(k_1+\dots+k_j+n_m+\tau_m+1-\tau_{j+1})\Gamma(\tau_{j+1}-n_j-\tau_j)}{\Gamma(k_1+\dots+k_j+n_m+\tau_m+1-n_j-\tau_j)} \\
&= \prod_{j=1}^{m-1}\frac{\Gamma(1+n_m+\tau_m-\tau_{j+1})\Gamma(\tau_{j+1}-n_j-\tau_j)}{\Gamma(1+n_m+\tau_m-n_j-\tau_j)} \\
&\sum_{k_1=0}^{n_1}\dots\sum_{k_{m-1}=0}^{n_{m-1}}\prod_{j=1}^{m-1}\frac{(-n_j)_{k_j}(1+n_m+\tau_m-\tau_{j+1})_{k_1+\dots+k_j}}{k_j!(1+n_m+\tau_m-n_j-\tau_j)_{k_1+\dots+k_j}}y^{k_j} \\
&= \prod_{j=1}^{m-1}\frac{(n_m+\tau_m-\tau_{j+1})!(\tau_{j+1}-n_j-\tau_j-1)!}{(n_m+\tau_m-n_j-\tau_j)!}Q_m(y).
\end{aligned}
$$

The inequality $0 \le Q_m(y) \le 1$ for $y \in [0,1]$ easily follows from the integral representation (10). □

Now, Theorem 4 is a consequence of Theorem 1 and Lemma 8.

7. Proof of Theorem 5

Setting $n_j = c_jn$, $j = 1,\dots,m$, $C = \sum_{j=1}^m c_j$, $\tau_1 = an+1$, $\tau_0 = bn+m$ in Corollary 4 we get that the absolute value of the remainder is less than

$$
\frac{M(\overline{c})}{r^{cn}}\int_0^1\int_0^1\frac{x^{(C+a)n+m}(1-x^r)^{Cn}t^{(C+a-b)n}(1-t)^{(b+c_m-C-a)n}\omega(t)}{(1-t+xt)^{c_mn+1}}\,dxdt
$$

with some constant $M(\overline{c}) < C^{m-1}$, since

$$
\prod_{j=1}^{m-1}\frac{Cn+j}{(c_{j+1}+\dots+c_m)n+m-j} < C^{m-1}.
$$

Denoting

$$f(x,t) = \frac{x^{C+a}(1-x^r)^C t^{C+a-b}(1-t)^{b+c_m-C-a}}{(1-t+xt)^{c_m}}$$

and applying Lemma 5 with $s = 1, d = c_m$, we conclude the theorem. □

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