

On a Conjecture of Erdős

T. Hessami Pilehrood* and K. Hessami Pilehrood**

Shahrekord University, Iran, Institute for Studies in Theoretical Physics and Mathematics (IPM)

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In this paper, we use the following standard notation: \mathbb{Z} is the ring of integers, \mathbb{Q} and $\overline{\mathbb{Q}}$ are the fields of rational and algebraic numbers, respectively, $\varphi(q)$ is the Euler function, and $\omega(q)$ is the number of different prime divisors of a number q .

In the 1960s, Erdős put forward the following conjecture [1, p. 430], which has not been either proved or disproved: *for any natural number q and any arbitrary numerical function $f(n)$ with period q taking the values $+1, -1$ for $n = 1, 2, \dots, q-1$ and 0 for $n = q$, the sum of the convergent series*

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

is nonzero. Note that the necessary and sufficient condition for the convergence of the series is the equality

$$\sum_{n=1}^q f(n) = \sum_{n=1}^{q-1} f(n) = 0,$$

implying that q is odd.

In 1969, Chowla posed the question of the existence of a periodic function $g: \mathbb{Z} \rightarrow \overline{\mathbb{Q}}$ with simple period q for which

$$\sum_{n=1}^{\infty} \frac{g(n)}{n} = 0.$$

A negative answer to this and even a more general question was given by Baker, Birch, and Wirsing in [2]. They proved that if a q -periodic, function $g: \mathbb{Z} \rightarrow \overline{\mathbb{Q}}$, not identically zero, satisfies the following conditions:

- 1) $g(r) = 0$ for $1 < (r, q) < q$;
- 2) the circular polynomial Φ_q is irreducible over $\mathbb{Q}(g(1), \dots, g(q))$, then the sum of the convergent series $\sum_{n=1}^{\infty} g(n)/n$ is nonzero.

In 1982, Okada [3], using [2], stated a theorem describing all q -periodic functions $g: \mathbb{Z} \rightarrow \overline{\mathbb{Q}}$ for which $\sum_{n=1}^{\infty} g(n)/n = 0$ under the condition that the circular polynomial Φ_q is irreducible over $\mathbb{Q}(g(1), \dots, g(q))$. The system of $\varphi(q) + \omega(q)$ homogeneous linear equations with rational coefficients in the unknowns $g(1), \dots, g(q)$ provides such a criterion. Using this criterion, a partial positive answer was given in [3] to the Erdős conjecture that the sum S is nonzero if $2\varphi(q) \geq q$, i.e., for all natural numbers q

*E-mail: hessamit@ipm.ir.

**E-mail: hessamik@ipm.ir.

such that $\omega(q) \leq 2$. Using [3], Tijdeman [4] proved the Erdős conjecture for all multiplicative functions f . Okada's result was improved by Saradha in [5], who showed that S is nonzero if $2\varphi(q) \geq q(1 - 1/h)$, where

$$h = \max_{1 \leq i \leq k} (p_i^{t_i}) \quad \text{and} \quad q = p_1^{t_1} \cdots p_k^{t_k}.$$

The question of the validity of the Erdős conjecture for $\omega(q) = 3$ and 4 was studied in the same paper.

Note that if g is a q -periodic function, then the sum of the convergent series

$$\sum_{n=1}^{\infty} g(n)/n$$

can be expressed as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{g(n)}{n} &= \sum_{k=0}^{\infty} \sum_{p=1}^q \frac{g(p)}{qk+p} = \sum_{k=0}^{\infty} \sum_{p=2}^q g(p) \left(\frac{1}{kq+p} - \frac{1}{kq+1} \right) \\ &= \frac{1}{q} \sum_{p=2}^q g(p) \left(\psi\left(\frac{1}{q}\right) - \psi\left(\frac{p}{q}\right) \right) = \frac{-1}{q} \sum_{p=1}^q g(p) \psi\left(\frac{p}{q}\right), \end{aligned} \tag{1}$$

where

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right)$$

is of the logarithmic derivative of the gamma function and $\gamma = -\psi(1)$ is the Euler constant. It readily follows from expression (1) that the Erdős conjecture is equivalent to the following statement: *any linear combination of the form*

$$\sum_{p=1}^{q-1} (\pm 1) \psi\left(\frac{p}{q}\right), \quad \text{where } q \text{ is an odd number,}$$

is nonzero. In contrast to this statement, the existence of a similar linear combination for an even q

$$\sum_{p=1}^{36} (\pm 1) \psi\left(\frac{p}{36}\right) = 0$$

was recently, proved by Tengely [6]. More precisely, he proved the following theorem.

Theorem 1. *There exists a function $f: \mathbb{N} \rightarrow \{-1, 1\}$ with period 36 for which*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

Tengely's proof is based on an extensive computer experiment and the finding of the following function:

$$\begin{aligned} f(n) &= 1, -1, -1, -1, -1, 1, 1, 1, -1, 1, -1, -1, 1, -1, 1, -1, -1, \\ &1, 1, 1, -1, 1, -1, -1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, 1 \end{aligned}$$

for $n = 1, 2, \dots, 36$, which satisfies Okada's criterion [3, Theorem 10]. Besides, he also showed that 36 is the minimal period of the function f , appearing in the theorem.

Here we present another elementary proof of Theorem 1 based on the following functional relation for the logarithmic derivative of the gamma function [7, Sec. 1.7.1]:

$$\psi(mz) = \frac{1}{m} \sum_{r=0}^{m-1} \psi\left(z + \frac{r}{m}\right) + \log m. \tag{2}$$

Proof of Theorem 1. Consider relation (2) for $m = 6$:

$$6\psi(6z) = \sum_{r=0}^5 \psi\left(z + \frac{r}{6}\right) + 6 \log 6. \tag{3}$$

Taking the difference of two identities obtained from (3) under the substitutions $z = 1/36$ and $z = 5/36$, we find

$$\sum_{r=0}^5 \left(\psi\left(\frac{1+6r}{36}\right) - \psi\left(\frac{5+6r}{36}\right) \right) = 6\psi\left(\frac{1}{6}\right) - 6\psi\left(\frac{5}{6}\right). \tag{4}$$

Treating relation (2) for $m = 3$ and successively substituting $z = 1/18$ and $z = 5/18$, we obtain

$$3\psi\left(\frac{1}{6}\right) - 3\psi\left(\frac{5}{6}\right) = \sum_{r=0}^2 \left(\psi\left(\frac{1+6r}{18}\right) - \psi\left(\frac{5+6r}{18}\right) \right). \tag{5}$$

Similarly, for $m = 2$, $z = 1/12$, and $z = 5/12$, from (2) we find

$$2\psi\left(\frac{1}{6}\right) - 2\psi\left(\frac{5}{6}\right) = \sum_{r=0}^1 \left(\psi\left(\frac{1+6r}{12}\right) - \psi\left(\frac{5+6r}{12}\right) \right). \tag{6}$$

Finally, from (4)–(6) we obtain

$$\begin{aligned} & \sum_{r=0}^5 \left(\psi\left(\frac{1+6r}{36}\right) - \psi\left(\frac{5+6r}{36}\right) \right) \\ &= \sum_{r=0}^2 \left(\psi\left(\frac{1+6r}{18}\right) - \psi\left(\frac{5+6r}{18}\right) \right) \\ & \quad + \sum_{r=0}^1 \left(\psi\left(\frac{1+6r}{12}\right) - \psi\left(\frac{5+6r}{12}\right) \right) + \psi\left(\frac{1}{6}\right) - \psi\left(\frac{5}{6}\right). \end{aligned} \tag{7}$$

Further, for $m = 2$, $z = 1/6$, and $z = 1/3$, from (2) we find

$$3\psi\left(\frac{1}{3}\right) - 3\psi\left(\frac{2}{3}\right) = \psi\left(\frac{1}{6}\right) - \psi\left(\frac{5}{6}\right). \tag{8}$$

On the other hand, substituting $m = 3$, $z = 1/9$, and $z = 2/9$ into (2), we can write

$$3\psi\left(\frac{1}{3}\right) - 3\psi\left(\frac{2}{3}\right) = \sum_{r=0}^2 \left(\psi\left(\frac{1+3r}{9}\right) - \psi\left(\frac{2+3r}{9}\right) \right). \tag{9}$$

Therefore, in view of (8), (9), we can rewrite (7) as

$$\begin{aligned} & \sum_{r=0}^5 \left(\psi\left(\frac{1+6r}{36}\right) - \psi\left(\frac{5+6r}{36}\right) \right) + \sum_{r=0}^2 \left(\psi\left(\frac{5+6r}{18}\right) - \psi\left(\frac{1+6r}{18}\right) \right) \\ & \quad + \sum_{r=0}^1 \left(\psi\left(\frac{5+6r}{12}\right) - \psi\left(\frac{1+6r}{12}\right) \right) + \sum_{r=0}^2 \left(\psi\left(\frac{2+3r}{9}\right) - \psi\left(\frac{1+3r}{9}\right) \right) = 0. \end{aligned} \tag{10}$$

Using the following nontrivial zero linear combination of the form:

$$\psi(1) - \psi\left(\frac{1}{3}\right) - \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{4}\right) - \psi\left(\frac{3}{4}\right) + \psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{6}\right) + \psi\left(\frac{5}{6}\right) = 0,$$

which can be obtained from (2) under an appropriate choice of $m = 2$, $z = 1/4$, $z = 1/2$ and $m = 3$, $z = 1/3$, $z = 1/6$, and adding it to (10), we obtain the required identity

$$\begin{aligned} & \sum_{r=0}^5 \left(\psi \left(\frac{1+6r}{36} \right) - \psi \left(\frac{5+6r}{36} \right) \right) + \sum_{r=0}^2 \left(\psi \left(\frac{5+6r}{18} \right) - \psi \left(\frac{1+6r}{18} \right) \right) \\ & + \sum_{r=0}^1 \left(\psi \left(\frac{5+6r}{12} \right) - \psi \left(\frac{1+6r}{12} \right) \right) + \sum_{r=0}^2 \left(\psi \left(\frac{2+3r}{9} \right) - \psi \left(\frac{1+3r}{9} \right) \right) \\ & + \psi \left(\frac{1}{6} \right) + \psi \left(\frac{5}{6} \right) - \psi \left(\frac{1}{4} \right) - \psi \left(\frac{3}{4} \right) - \psi \left(\frac{1}{3} \right) - \psi \left(\frac{2}{3} \right) + \psi \left(\frac{1}{2} \right) + \psi(1) = 0. \end{aligned}$$

□

In conclusion, note that, as shown in [8, Theorem 1], all the infinite nonzero sums examined in the present paper are transcendental numbers, which also easily follows from expression (1), the Gauss formula for the logarithmic derivative of the gamma function [9, Sec. 1.2, p. 14]

$$\psi \left(\frac{p}{q} \right) = -\gamma - \log q + \sum_{k=1}^{q-1} \omega^{-kp} \log(1 - \omega^k), \quad \text{where } p, q \in \mathbb{Z}, 0 < p < q, \omega = e^{2\pi i/q},$$

and Baker's theory of linear forms in the logarithms of algebraic numbers.

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