INFINITE SUMS AS LINEAR COMBINATIONS OF POLYGAMMA FUNCTIONS

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Dedicated to Professor Yu. V. Nesterenko
on the occasion of his 60th birthday

Abstract. We consider the series

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}f(n),$$

where $P(x), Q(x)$ are polynomials with algebraic coefficients satisfying some symmetry conditions, f is a number theoretic periodic function taking algebraic values, and all the roots of the polynomial $Q(x)$ lie in a fixed imaginary quadratic field. Summing the series S explicitly and applying famous Nesterenko's result on algebraic independence of the numbers $\pi, e^{\pi\sqrt{d}}$ we show that the infinite sum S either has a computable algebraic value or is transcendental. Similar assertions for certain trigonometric series are obtained. For more general set of roots of the polynomial $Q(x)$ we give some statements on the transcendence of the sum S provided that the Schanuel conjecture holds.

1. Introduction

1. We begin with some notations and definitions. Let d be a positive square-free integer. We denote by $\mathbb{Z}, \mathbb{Q}, \overline{\mathbb{Q}}$, and $\mathbb{Q}(i\sqrt{d})$ the set of integers, the field of rational numbers, the field of algebraic numbers, and an imaginary quadratic field, respectively.

We will use the polygamma function

$$\psi^{(k)}(z)=\frac{d^k}{dz^k}\psi(z)=\frac{d^{k+1}}{dz^{k+1}}\log\Gamma(z), \qquad k=1,2,\dots,$$

which has the following series expansion (see [2], §1.16):

$$\psi^{(k)}(z)=(-1)^{k+1}k!\sum_{n=0}^{\infty}\frac{1}{(n+z)^{k+1}}, \qquad z\neq 0,-1,-2,\dots, \tag{1}$$

and the logarithmic derivative of $\Gamma(z)$

$$\psi(z)=\frac{d}{dz}\log\Gamma(z)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right), \qquad z\neq 0,-1,-2,\dots,$$

which denotes the digamma function. Obviously, $\psi(1)=-\gamma$, where γ is Euler's constant. The function $\psi^{(k)}(z)$, $k=0,1,2,\dots$, is single valued

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and analytic in the whole complex plane except for the points $z=-m$, $m=0,1,2,\dots$, where it possesses poles of order $(k+1)$. The polygamma function satisfies many functional relations [2, §1.16] such as

"recurrence formula":

$$\psi^{(k)}(z+1)=\psi^{(k)}(z)+\frac{(-1)^k k!}{z^{k+1}}, \tag{2}$$

"reflection formula":

$$\psi^{(k)}(1-z)+(-1)^{k+1}\psi^{(k)}(z)=(-1)^k\pi\frac{d^k}{dz^k}\cot\pi z, \tag{3}$$

"multiplication formula":

$$\psi^{(k)}(mz)=\delta\log m+\frac{1}{m^{k+1}}\sum_{r=0}^{m-1}\psi^{(k)}\Bigl(z+\frac{r}{m}\Bigr),$$

where $\delta=1$ if $k=0$ and $\delta=0$ if $k>0$.

We also introduce its alternating analog (see [2, §1.16])

$$g^{(k)}(z)=(-1)^k k!\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+z)^{k+1}}=\frac{1}{2^{k+1}}\left(\psi^{(k)}\Bigl(\frac{z+1}{2}\Bigr)-\psi^{(k)}\Bigl(\frac{z}{2}\Bigr)\right), \tag{4}$$

which satisfies the similar functional relations

$$g^{(k)}(z+1)=\frac{(-1)^k k!}{z^{k+1}}-g^{(k)}(z), \qquad k=1,2,\dots, \tag{5}$$

$$g^{(k)}(z)+(-1)^k g^{(k)}(1-z)=\pi\frac{d^k}{dz^k}\Bigl(\frac{1}{\sin\pi z}\Bigr). \tag{6}$$

Obviously by (1) and (4), we have that the numbers $\psi^{(k)}(1)/\zeta(k+1)$, $g^{(k)}(1)/\zeta(k+1)$, $\psi^{(k)}(1/2)/\zeta(k+1)$ are rational (here $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function) and therefore from (2), (5) we get the following inclusions:

$$\psi^{(2k-1)}(m), \quad g^{(2k-1)}(m), \quad \psi^{(2k-1)}\Bigl(m+\frac{1}{2}\Bigr)\in\mathbb{Q}^{\times}\cdot\pi^{2k}+\mathbb{Q}, \quad m\in\mathbb{N}. \tag{7}$$

2. In the paper, we consider the values of the series

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}, \qquad T=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}(-1)^n, \qquad U=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}f(n), \tag{8}$$

where $P(x),Q(x)\in\overline{\mathbb{Q}}[x]$ and f is a periodic number theoretic function, and express them as linear combinations of values of the polygamma functions (see Lemmas 1–2 below). Such a representation allows one to give simple sufficient conditions for the numbers S,T to be algebraic or transcendental that is done in section 2. Further, we assume that all the zeros of $Q(x)$ are in the imaginary quadratic field $\mathbb{Q}(i\sqrt{d})$ and the polynomials $P(x),Q(x)$ possess some symmetry properties. By formulas (3), (6), summing the series S,T,U explicitly and applying famous Nesterenko's result [7] on algebraic independence of the numbers $\pi, e^{\pi\sqrt{d}}$ we show that the infinite sums (8) either have a computable algebraic value or are transcendental. (By a computable value, we mean a number which can be explicitly determined in terms of its defining parameters.) Actually, we describe double approach for computation of infinite sums (8) combining linear combinations of values of the polygamma functions and contour integration. The latter one can be applied to the trigonometric series

$$V=\sum_{n=-\infty}^{+\infty}\frac{P_1(n)e^{i\beta_1 n}+\ldots+P_s(n)e^{i\beta_s n}}{Q(n)},\qquad \beta_1,\ldots,\beta_s\in\mathbb{Q}$$

that enables us to prove that under certain conditions on the polynomials $P_1,\ldots,P_s,Q$, the sum V is either zero or transcendental. As a consequence, we establish the transcendence of some Fourier series (see section 4). In section 5 we extend these results to a more general set of roots of the polynomial $Q(x)$ provided that the Schanuel conjecture holds. This generalizes the well-known result of P. Bundschuh on the series $\sum_{n=2}^{\infty}\frac{1}{n^{2k}-1}$, $k\geq 2$ (see [3], [12, section 3.2]).

Special cases of the infinite sums (8) were considered by P. Bundschuh in [3]. Using Baker's theory on linear forms in logarithms, he proved that the value of the series

$$F(z)=z\cdot\sum_{m=1}^{\infty}\frac{a_m}{m(m-z)},$$

where $\{a_m\}_{m=1}^{\infty}$ is a periodic sequence of algebraic numbers and $z\in\mathbb{Q}\cap(0,1)$, is either zero or transcendental. In particular, this yields the transcendence of the numbers $\psi(z)+\gamma$, $\psi(z)-\psi(z/2)$ for any $z\in\mathbb{Q}\setminus\mathbb{Z}$, and the series $\sum_{n=2}^{\infty}\zeta(n)z^n$, $\sum_{n=2}^{\infty}\beta(n)z^n$ for any rational z with $0<|z|<1$, where $\beta(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}$ is the Dirichlet beta function.

The case when all the roots $\alpha_1,\ldots,\alpha_m$ of $Q(x)$ are distinct rational numbers was considered in [1], where by Baker's theory it was proved that each of the numbers (8) is equal to a computable algebraic number or transcendental. In particular, if $Q(x)$ is a reduced polynomial, i.e., if $\alpha_1,\ldots,\alpha_m$ are distinct rational numbers from $[-1,0)$, then S,T,U and the series

$$\sum_{n=0}^{\infty}\frac{P_1(n)\beta_1^n+\ldots+P_s(n)\beta_s^n}{Q(n)},\qquad \beta_1,\ldots,\beta_s\in\overline{\mathbb{Q}}$$

are either zero or transcendental.

Let us notice that from work [1] it follows that for any rational numbers $\alpha_1,\ldots,\alpha_m$ distinct from zero and negative integers and such that $\alpha_k-\alpha_l\notin\mathbb{Z}$, $1\leq k\neq l\leq m$, all the values

$$\psi(\alpha_1),\ldots,\psi(\alpha_m)\tag{9}$$

are transcendental except for at most one value of α_k. (Compare it with [6, theorem 3]). In fact, taking into account (2) we can assume without loss of generality that $\alpha_1, \dots, \alpha_m$ are distinct numbers from $(0,1]$ and then by [1, theorem 3] we have for $k \neq l$

$$\psi(\alpha_l) - \psi(\alpha_k) = \sum_{n=0}^{\infty} \left(\frac{1}{n+\alpha_k} - \frac{1}{n+\alpha_l} \right) = \sum_{n=0}^{\infty} \frac{\alpha_l - \alpha_k}{(n+\alpha_k)(n+\alpha_l)} \notin \overline{\mathbb{Q}}.$$

Therefore the set (9) cannot contain two algebraic numbers.

In 2001, G. Molteni [5] considered the generating power series for the sequence $\{\zeta(2k+1)\}_{k=1}^{\infty}$, which also can be written as a linear combination of values of the digamma function

$$F(z) = \sum_{k=1}^{\infty} \zeta(2k+1) z^{2k} = -\frac{1}{2}\psi(1+z) - \frac{1}{2}\psi(1-z) + \psi(1)$$

and proved that the numbers $1, F(\alpha_1), \dots, F(\alpha_m)$ are linearly independent over $\overline{\mathbb{Q}}$ if all $\alpha_k = \frac{a_k}{b_k}$ are distinct rational numbers from the interval $(0,1)$ such that $(a_k, b_k) = 1$ and for any k there exists an odd prime p_k dividing b_k and $p_k \nmid b_j$ when $j \neq k$. An obvious corollary is that $F(\alpha)$ is transcendental for arbitrary $\alpha = a/b \in (0,1)$ with b not a power of 2. Actually, this restriction can be removed and $F(\alpha)$ is transcendental for any rational α with $0 < |\alpha| < 1$ by [1, theorem 3], since

$$F(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^2}{(n+1)(n+1+\alpha)(n+1-\alpha)}$$

and the last series does not vanish.

2. Sums S, T, U as linear combinations of polygamma functions

Lemma 1. *Let $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ be periodic with period $q \in \mathbb{N}$. Suppose that* $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $\deg P(x) \le \deg Q(x) - 1$, *and* $Q(x) = (x+\alpha_1)^{l_1} \dots (x+\alpha_m)^{l_m}$, *where $l_1, \dots, l_m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m$ are distinct, and distinct from zero and the negative integers. If* $\deg P(x) = \deg Q(x) - 1$, *suppose also that $\sum_{t=0}^{q-1} f(t) = 0$ (convergence condition). Then the series*

$$U = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} f(n)$$

converges and we have the following representation:

$$U = \sum_{t=0}^{q-1} f(t) \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!} \frac{A_{k,l}}{q^l} \psi^{(l-1)}\Big(\frac{t+\alpha_k}{q}\Big) \tag{10}$$

with

$$A_{k,l} = \frac{1}{(l_k - l)!} \frac{d^{l_k - l}}{dx^{l_k - l}} \left(\frac{P(x)}{Q(x)} (x+\alpha_k)^{l_k} \right)\Bigg|_{x=-\alpha_k} \in \overline{\mathbb{Q}}. \tag{11}$$

Proof. Writing n in the form $n = q\tau + t$, $\tau, t \in \mathbb{Z}$, $0 \le t \le q-1$, $\tau \ge 0$, we get

$$U = \sum_{\tau=0}^{\infty}\sum_{t=0}^{q-1} f(q\tau+t)\frac{P(q\tau+t)}{Q(q\tau+t)} = \sum_{\tau=0}^{\infty}\sum_{t=0}^{q-1} f(t)\frac{P(q\tau+t)}{Q(q\tau+t)}. \tag{12}$$

Decomposing $P(x)/Q(x)$ into partial fractions, we have

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}}{(x+\alpha_k)^l},$$

where the coefficients $A_{k,l}$ are defined in (11) and $\sum_{k=1}^{m} A_{k,1} = 0$ if $\deg P(x) \le \deg Q(x) - 2$.

To prove (10), we first suppose that $\deg P(x) \le \deg Q(x) - 2$. Then from (12) we have

$$U = \sum_{t=0}^{q-1} f(t)\sum_{\tau=0}^{\infty}\frac{P(q\tau+t)}{Q(q\tau+t)},$$

where

$$\begin{aligned}\frac{P(q\tau+t)}{Q(q\tau+t)} &= \sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l} = \sum_{k=1}^{m}\frac{A_{k,1}}{q\tau+t+\alpha_k} + \sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}\\ &= \frac{1}{q}\sum_{k=2}^{m}A_{k,1}\Big(\frac{1}{\tau+\frac{t+\alpha_k}{q}} - \frac{1}{\tau+\frac{t+\alpha_1}{q}}\Big) + \sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\tau=0}^{\infty}\frac{P(q\tau+t)}{Q(q\tau+t)} &= \frac{1}{q}\sum_{k=2}^{m}A_{k,1}\left(\psi\Big(\frac{t+\alpha_1}{q}\Big) - \psi\Big(\frac{t+\alpha_k}{q}\Big)\right)\\ &+ \sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{(-1)^l}{(l-1)!}\frac{A_{k,l}}{q^l}\psi^{(l-1)}\Big(\frac{t+\alpha_k}{q}\Big) = \sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\frac{A_{k,l}}{q^l}\psi^{(l-1)}\Big(\frac{t+\alpha_k}{q}\Big),\end{aligned}$$

which yields (10). If $\deg P(x) = \deg Q(x) - 1$, then we find

$$\begin{aligned}\sum_{t=0}^{q-1}\frac{P(q\tau+t)}{Q(q\tau+t)}f(t) &= \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}\\ &= \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\frac{A_{k,1}}{q\tau+t+\alpha_k} + \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}\\ &= \sum_{k=1}^{m}\frac{A_{k,1}}{q}\sum_{t=1}^{q-1}f(t)\Big(\frac{1}{\tau+\frac{t+\alpha_k}{q}} - \frac{1}{\tau+\frac{\alpha_k}{q}}\Big) + \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{A_{k,l}}{(q\tau+t+\alpha_k)^l}.\end{aligned}$$

Hence, by (12), we get

$$\begin{aligned}U &= \sum_{k=1}^{m}\frac{A_{k,1}}{q}\sum_{t=1}^{q-1}f(t)\left(\psi\Big(\frac{\alpha_k}{q}\Big) - \psi\Big(\frac{t+\alpha_k}{q}\Big)\right) + \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=2}^{l_k}\frac{(-1)^l}{(l-1)!}\frac{A_{k,l}}{q^l}\\ &\times \psi^{(l-1)}\Big(\frac{t+\alpha_k}{q}\Big) = \sum_{t=0}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\frac{A_{k,l}}{q^l}\psi^{(l-1)}\Big(\frac{t+\alpha_k}{q}\Big),\end{aligned}$$

as required. $\square$

Let us mention two particular cases $q=1$, $f\equiv 1$ and $q=2$, $f(n)=(-1)^n$ of Lemma 1.

Lemma 2. *Let* $P(x), Q(x)\in\overline{\mathbb{Q}}[x]$, $Q(x)=(x+\alpha_1)^{l_1}\dots(x+\alpha_m)^{l_m}$, *where* $l_1,\dots,l_m\in\mathbb{N}$ *and* $\alpha_1,\dots,\alpha_m$ *are distinct, and distinct from zero and the negative integers. Suppose that the series*

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}, \qquad T=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}(-1)^n$$

converge. Then the following representations are valid:

$$S=\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}A_{k,l}\psi^{(l-1)}(\alpha_k), \qquad T=\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^{l-1}}{(l-1)!}A_{k,l}g^{(l-1)}(\alpha_k),$$

where the coefficients $A_{k,l}$ *are defined in* (11).

If $Q(x)$ has only simple zeros, then Lemma 2 enables us to give simple sufficient conditions for S, T to be algebraic or transcendental.

Corollary 1. *Let* $P(x), Q(x)\in\overline{\mathbb{Q}}[x]$, $Q(x)=(x+\alpha_1)\dots(x+\alpha_m)$, *where* $\alpha_1,\dots,\alpha_m$ *are distinct, and distinct from zero and the negative integers, and* $\deg P(x)\le\deg Q(x)-2$. *If there is a subset* L *of* $\{1,2,\dots,m\}$ *with* $\#L\ge 2$, *with* $j,k\in L\Rightarrow \alpha_j-\alpha_k\in\mathbb{Z}$, *and with* $P(-\alpha_l)=0$ *for* $l\notin L$, *then*

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}$$

is algebraic.

Proof. This statement easily follows from Lemma 2 and formula (2). $\square$

Remark 0.1. In the case $m=3$ and $\alpha_1,\dots\alpha_m\in\mathbb{Q}$, $P(x),Q(x)\in\mathbb{Q}[x]$ the conditions of Corollary 1 are necessary and sufficient for S to be rational (see [9, Theorem 2]).

Corollary 2. *Let* $P(x), Q(x)\in\overline{\mathbb{Q}}[x]$, $Q(x)=(x+\alpha_1)\dots(x+\alpha_m)$, *where* $\alpha_1,\dots,\alpha_m$ *are distinct, and distinct from zero and the negative integers, and* $\deg P(x)\le\deg Q(x)-1$. *If* $\alpha_k-\alpha_1=: n_k\in\mathbb{Z}$ *for all* $1\le k\le m$ *and*

$$\sum_{k=1}^{m}(-1)^{n_k}\frac{P(-\alpha_k)}{Q'(-\alpha_k)}=0, \tag{13}$$

then

$$T=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}(-1)^n$$

is algebraic. (In particular, if all n_k *are even and* $\deg P(x)\le\deg Q(x)-2$, *then condition* (13) *holds automatically).*

Proof. This statement easily follows from Lemma 2 and formula (5). $\square$

Remark 0.2. In the case $m = 2$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$, $P(x), Q(x) \in \mathbb{Q}[x]$ the conditions of Corollary 2 are necessary and sufficient for T to be rational (see [9, Theorem 1] and [10, Theorem 3]).

Corollary 3. *Let* $P(x) \in \overline{\mathbb{Q}}[x]$, $Q(x) = (x + \alpha_1) \ldots (x + \alpha_m)$, *where* $\alpha_1, \ldots, \alpha_m$ *are distinct rational numbers, distinct from zero and the negative integers, and* $\deg P(x) = m - 1$. *If* $\alpha_k - \alpha_l \in 2\mathbb{Z}$ *for all* $1 \leq k, l \leq m$, *then the sum*

$$T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} (-1)^n$$

is transcendental.

Proof. By Lemma 2 and formula (5) it follows that

$$T = A + a \cdot g(\alpha_1) = B \pm a \cdot g(\alpha) = B \pm a \cdot \sum_{n=0}^{\infty} \Big(\frac{1}{2n + \alpha} - \frac{1}{2n + \alpha + 1} \Big),$$

where $A, B \in \overline{\mathbb{Q}}$, $a \neq 0$ is the leading coefficient of the polynomial $P(x)$ and $\alpha \equiv \alpha_1 \pmod 1$, $\alpha \in (0, 1]$. Since the infinite sum in the latter expression of T does not vanish, by [1, Theorem 3] we conclude that T is transcendental. $\square$

Lemma 3. *For the* k*-th derivatives we have*

$$(a) \quad (\cot \pi z)^{(k)} = \pi^k \cdot p_k(\cot \pi z), \qquad (b) \quad \left(\frac{1}{\sin \pi z} \right)^{(k)} = \pi^k \cdot \frac{q_k(\cos \pi z)}{\sin^{k+1} \pi z},$$

where $p_k(z), q_k(z) \in \mathbb{Z}[z]$, $\deg(p_k(z) - (-1)^k k! z^{k+1}) \leq k$, $\deg(q_k(z) - (-z)^k) \leq k - 1$.

Proof. The proof is by induction on k. Obviously, for $k = 0$ formulas $(a), (b)$ are valid with $p_0(z) = z$ and $q_0(z) = 1$. Assuming $(a), (b)$ to hold for k, we will prove them for $k + 1$. We have

$$(\cot \pi z)^{(k+1)} = \pi^k \cdot (p_k(\cot \pi z))' = \pi^{k+1} \cdot p_{k+1}(\cot \pi z),$$

where $p_{k+1}(z) = -p_k'(z) \cdot (z^2 + 1) = (-1)^{k+1}(k + 1)! z^{k+2} + c_{k+1} z^{k+1} + \ldots \in \mathbb{Z}[z]$, and

$$\left(\frac{1}{\sin \pi z} \right)^{(k+1)} = \pi^k \cdot \left(\frac{q_k(\cos \pi z)}{\sin^{k+1} \pi z} \right)' = \pi^{k+1} \cdot \frac{q_{k+1}(\cos \pi z)}{\sin^{k+2} \pi z}$$

with $q_{k+1}(z) = q_k'(z) \cdot (z^2 - 1) - (k + 1) z \cdot q_k(z) = (-1)^{k+1} z^{k+1} + d_k z^k + \ldots \in \mathbb{Z}[z]$. $\square$

3. Main results

Theorem 1. *Let* $P_1,\dots,P_s,\ Q_1,\dots,Q_s\in\overline{\mathbb{Q}}[x],\ m_1,\dots,m_s\in\mathbb{N},\ r_1,\dots,r_s\in\mathbb{Z}$ *satisfy the following conditions: for any* $1\le j\le s$, $\deg P_j\le\deg Q_j-2$,

$$\frac{P_j(-x)}{Q_j(-x)}=\frac{P_j(r_j+x)}{Q_j(r_j+x)} \tag{14}$$

$Q_j(x)=\prod_{k=1}^{2m_j}(x-\alpha_{j,k})^{l_{j,k}}$, *where* $\alpha_{j,k}=a_{j,k}+ib_{j,k}\sqrt{d}\in\mathbb{Q}(i\sqrt{d})\setminus\mathbb{N}_0$, $k=1,\dots,2m_j$, *are distinct and such that* $\alpha_{j,m_j+k}=r_j-\alpha_{j,k}$, $b_{j,k}\ge 0$, $l_{j,m_j+k}=l_{j,k}\in\mathbb{N}$, $k=1,2,\dots,m_j$. *Then the sum*

$$S=\sum_{n=0}^{\infty}\Big(\frac{P_1(n)}{Q_1(n)}+\ldots+\frac{P_s(n)}{Q_s(n)}\Big)$$

is either a computable algebraic number or transcendental. Moreover, S *is transcendental if at least one of the following conditions holds:*

(i) $\alpha_{j,k}\notin\mathbb{Q}\backslash\mathbb{Z}$, $j=1,\dots,s$, $k=1,\dots,2m_j$, *and* $\displaystyle\sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{m_j}\operatorname*{res}_{z=\alpha_{j,k}}\frac{P_j(z)}{Q_j(z)}\ne 0,$

(ii) $b_{j_0,k_0}:=\min\{b_{j,k}:b_{j,k}>0\}$ *is a unique minimum of the positive numbers* $b_{j,k}$ *and* $\operatorname*{res}_{z=\alpha_{j_0,k_0}}\frac{P_{j_0}(z)}{Q_{j_0}(z)}\ne 0,$

(iii) there exists a unique maximum l_{j_0,k_0} *of the sequence* $l_{j,k}$, $1\le j\le s$, $1\le k\le m_j$, *and* $b_{j_0,k_0}>0$, $P_{j_0}(\alpha_{j_0,k_0})\ne 0$.

Proof. By Lemma 2, we have

$$S=\sum_{j=1}^{s}\sum_{n=0}^{\infty}\frac{P_j(n)}{Q_j(n)}=\sum_{j=1}^{s}\sum_{k=1}^{2m_j}\sum_{l=1}^{l_{j,k}}\frac{(-1)^l}{(l-1)!}A_{j,k,l}\cdot\psi^{(l-1)}(-\alpha_{j,k}),$$

where

$$A_{j,k,l}=\frac{1}{(l_{j,k}-l)!}\Big(\frac{d}{dx}\Big)^{l_{j,k}-l}\left(\frac{P_j(x)}{Q_j(x)}(x-\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{x=\alpha_{j,k}}\in\overline{\mathbb{Q}}. \tag{15}$$

From (14), (15) for $1\le k\le m_j$ it follows that

$$A_{j,m_j+k,l}=\frac{1}{(l_{j,k}-l)!}\Big(\frac{d}{dx}\Big)^{l_{j,k}-l}\left(\frac{P_j(r_j-x)}{Q_j(r_j-x)}(x-r_j+\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{x=r_j-\alpha_{j,k}}$$

$$=\frac{(-1)^l}{(l_{j,k}-l)!}\Big(\frac{d}{dy}\Big)^{l_{j,k}-l}\left(\frac{P_j(y)}{Q_j(y)}(y-\alpha_{j,k})^{l_{j,k}}\right)\bigg|_{y=\alpha_{j,k}}=(-1)^l\cdot A_{j,k,l}$$

with $y=r_j-x$. Therefore,

$$S=\sum_{j=1}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}}\frac{(-1)^l}{(l-1)!}A_{j,k,l}\Big(\psi^{(l-1)}(-\alpha_{j,k})+(-1)^l\psi^{(l-1)}(\alpha_{j,k}-r_j)\Big).$$

Now if for some pair (j,k) we have $-\alpha_{j,k}$ and $\alpha_{j,k}-r_j\in\mathbb{N}$, then by (2), (7), we get

$$S=C_0+\sum_{\substack{j=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{\substack{l=1\\ l\,even}}^{l_{j,k}}C_{j,k,l}\pi^l+\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}}\frac{A_{j,k,l}}{(l-1)!}\Big(\psi^{(l-1)}(\alpha_{j,k}+1)$$
$$+(-1)^l\psi^{(l-1)}(-\alpha_{j,k})\Big),$$

where $C_0,C_{j,k,l}\in\overline{\mathbb{Q}}$. Combining this with (3) and Lemma 3 we conclude that

(16)
$$S=C_0+\sum_{\substack{j=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{\substack{l=1\\ l\,even}}^{l_{j,k}}C_{j,k,l}\pi^l+\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}\sum_{l=1}^{l_{j,k}}\frac{(-1)^{l-1}A_{j,k,l}}{(l-1)!}\pi^l\cdot p_{l-1}(-\cot\pi\alpha_{j,k}).$$

According to the formula

$$\cot(\pi\alpha_{j,k})=i\frac{e^{2\pi i a_{j,k}}+e^{2\pi b_{j,k}\sqrt{d}}}{e^{2\pi i a_{j,k}}-e^{2\pi b_{j,k}\sqrt{d}}}=-i-\frac{2ie^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}$$

we get that $S-C_0\in\overline{\mathbb{Q}}(\pi,e^{\frac{\pi\sqrt{d}}{B}})$, where $B\in\mathbb{N}$ is the least common denominator of the numbers $b_{j,k}$, and therefore, $S-C_0$ is either zero or transcendental in view of the algebraic independence of π and $e^{\pi\sqrt{d}}$ [7].

If we suppose that S is algebraic and condition (i) holds, then considering the summands in (16) involving π to the first power we get

$$-\pi\sum_{\substack{j=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{s}\sum_{k=1}^{m_j}A_{j,k,1}\cot\pi\alpha_{j,k}+\pi^2(\ldots)=0$$

or

$$\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}A_{j,k,1}+2\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}\frac{A_{j,k,1}e^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}+\pi^2(\ldots)=0.$$

Now multiplying both sides of the last equality by

$$\prod_{\substack{j=1\\ b_{j,k}>0}}^{s}\prod_{k=1}^{m_j}(e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}})^{l_{j,k}} \tag{17}$$

we get a contradiction with the algebraic independence of π and $e^{\pi\sqrt{d}}$.

If (ii) is valid and S is algebraic, then (16) can be rewritten as

$$\pi C_1+2\pi i\sum_{\substack{j=1\\ b_{j,k}>0}}^{s}\sum_{k=1}^{m_j}\frac{A_{j,k,1}e^{2\pi i a_{j,k}}}{e^{2\pi b_{j,k}\sqrt{d}}-e^{2\pi i a_{j,k}}}+\pi^2(\ldots)=0. \tag{18}$$

If $C_1 \neq 0$, then it is impossible by the same argument as above. If $C_1 = 0$, then multiplying both sides of (18) by (17) we get

$$2\pi i A_{j_0,k_0,1} e^{2\pi i a_{j_0,k_0}} e^{2\pi(\beta - b_{j_0,k_0})\sqrt{d}} + \ldots = 0,$$

which is impossible, therefore, S is transcendental.

If condition (*iii*) holds, then the summands with the maximal power of π in (16) have the form

$$\pi^{l_{j_0,k_0}} \Big(\pm \frac{A_{j_0,k_0,l_{j_0,k_0}}}{(l_{j_0,k_0}-1)!} \cdot p_{l_{j_0,k_0}-1}(-\cot \pi\alpha_{j_0,k_0}) + C_{j_0,k_0,l_{j_0,k_0}} \Big), \tag{19}$$

where $A_{j_0,k_0,l_{j_0,k_0}}, C_{j_0,k_0,l_{j_0,k_0}} \in \overline{\mathbb{Q}}$ and $A_{j_0,k_0,l_{j_0,k_0}} \neq 0$ by (15). Since $\cot \pi\alpha_{j_0,k_0}$ is transcendental, the term (19) doesn't vanish in (16) and hence, S is transcendental. This completes the proof of the theorem. $\square$

Remark 1.1. If under the assumptions of Theorem 1 we have $r_1 = \ldots = r_s = -1$, then S is either zero or transcendental.

Corollary 4. *If* $a, b \in \mathbb{Z}$, $4b > a^2$, $m \in \mathbb{N}$, *then the sum*

$$\sum_{n=0}^{\infty} \frac{P(n)}{(n^2+an+b)^m}$$

is transcendental for any polynomial $P(x) \in \overline{\mathbb{Q}}[x]$ *such that* $\deg P(x) \leq 2m-2$ *and* $P(-x) = P(x-a)$. *In particular, the sum of the series*

$$\sum_{n=0}^{\infty} \frac{(n^2+an+c)^k}{(n^2+an+b)^m}$$

is transcendental for any $c, k \in \mathbb{Z}$, $0 \leq k < m$.

Theorem 2. *Let* $P_1, \ldots, P_s$, $Q_1, \ldots, Q_s \in \overline{\mathbb{Q}}[x]$, $m_1, \ldots, m_s \in \mathbb{N}$, $r_1, \ldots, r_s \in \mathbb{Z}$ *satisfy the following conditions: for any* $1 \leq j \leq s$, $\deg P_j \leq \deg Q_j - 1$,

$$\frac{P_j(-x)}{Q_j(-x)} = (-1)^{r_j} \frac{P_j(r_j+x)}{Q_j(r_j+x)}. \tag{20}$$

$Q_j(x) = \prod_{k=1}^{2m_j} (x - \alpha_{j,k})^{l_{j,k}}$, *where* $\alpha_{j,k} = a_{j,k} + i b_{j,k}\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}_0$, $k = 1, \ldots, 2m_j$, *are distinct and such that* $\alpha_{j,m_j+k} = r_j - \alpha_{j,k}$, $b_{j,k} \geq 0$, $l_{j,m_j+k} = l_{j,k} \in \mathbb{N}$, $k = 1, 2, \ldots, m_j$. *Then the sum*

$$T = \sum_{n=0}^{\infty} \Big(\frac{P_1(n)}{Q_1(n)} + \ldots + \frac{P_s(n)}{Q_s(n)} \Big) (-1)^n$$

is either a computable algebraic number or transcendental. Moreover, T *is transcendental if at least one of the following conditions holds:*

(*i*) $b_{j_0,k_0} := \min\{b_{j,k} : b_{j,k} > 0\}$ *is a unique minimum of the positive numbers* $b_{j,k}$ *and* $\operatorname*{res}_{z=\alpha_{j_0,k_0}} \frac{P_{j_0}(z)}{Q_{j_0}(z)} \neq 0$,

(*ii*) *there exists a unique maximum* l_{j_0,k_0} *of the sequence* $l_{j,k}$, $1 \leq j \leq s$,

$1 \le k \le m_j, \quad$ *and* $\quad b_{j_0,k_0} > 0,\ P_{j_0}(\alpha_{j_0,k_0}) \neq 0.$

Proof. From Lemma 2 it follows that

$$T = \sum_{j=1}^{s} \sum_{n=0}^{\infty} \frac{P_j(n)}{Q_j(n)}(-1)^n = \sum_{j=1}^{s} \sum_{k=1}^{2m_j} \sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}}{(l-1)!} A_{j,k,l}\, g^{(l-1)}(-\alpha_{j,k}),$$

where the coefficients $A_{j,k,l}$ are defined in (15). According to (15) and (20) for $1 \le k \le m_j$ we have $A_{j,m_j+k,l} = (-1)^{r_j+l} \cdot A_{j,k,l}$. Then

$$T = \sum_{j=1}^{s} \sum_{k=1}^{m_j} \sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}}{(l-1)!} A_{j,k,l} \Big(g^{(l-1)}(-\alpha_{j,k}) + (-1)^{r_j+l} g^{(l-1)}(\alpha_{j,k} - r_j) \Big).$$

Now if for some pair (j,k) we have $-\alpha_{j,k}$ and $\alpha_{j,k} - r_j \in \mathbb{N}$, then by (5), (7), we get

$$T = C_0 + \sum_{\substack{j=1 \\ \alpha_{j,k} \in \mathbb{Z}}}^{s} \sum_{k=1}^{m_j} \sum_{\substack{l=1 \\ l\, even}}^{l_{j,k}} C_{j,k,l} \pi^l + \sum_{\substack{j=1 \\ \alpha_{j,k} \notin \mathbb{Z}}}^{s} \sum_{k=1}^{m_j} \sum_{l=1}^{l_{j,k}} \frac{A_{j,k,l}}{(l-1)!} \Big((-1)^{l-1} g^{(l-1)}(-\alpha_{j,k})$$

$$+ g^{(l-1)}(\alpha_{j,k} + 1) \Big),$$

where $C_0, C_{j,k,l} \in \overline{\mathbb{Q}}$. Hence, by (6) and Lemma 3, we have

$$T = C_0 + \sum_{\substack{j=1 \\ \alpha_{j,k} \in \mathbb{Z}}}^{s} \sum_{k=1}^{m_j} \sum_{\substack{l=1 \\ l\, even}}^{L} C_{j,k,l} \pi^l - \sum_{\substack{j=1 \\ \alpha_{j,k} \notin \mathbb{Z}}}^{s} \sum_{k=1}^{m_j} \sum_{l=1}^{l_{j,k}} \frac{A_{j,k,l}}{(l-1)!} \pi^l \frac{q_{l-1}(\cos \pi \alpha_{j,k})}{\sin^l(\pi \alpha_{j,k})} \tag{21}$$

and according to Euler's formulas for cos and sin we conclude that either $T = C_0$ or T is transcendental.

If T is algebraic and condition (i) holds, then we rewrite (21) as

$$\pi C_1 + \pi \sum_{\substack{j=1 \\ b_{j,k} > 0}}^{s} \sum_{k=1}^{m_j} \frac{A_{j,k,1}}{\sin(\pi \alpha_{j,k})} + \pi^2(\ldots) = 0,$$

from which by the same argument as in the proof of Theorem 1 (ii) and formula

$$\frac{1}{\sin(\pi \alpha_{j,k})} = -\frac{2i e^{i\pi a_{j,k}} e^{\pi b_{j,k}\sqrt{d}}}{e^{2\pi b_{j,k}\sqrt{d}} - e^{2\pi i a_{j,k}}}$$

we get a contradiction.

If condition (ii) is valid and T is algebraic, then from (21) we have

$$\pi^{l_{j_0,k_0}} \left(C_{j_0,k_0,l_{j_0,k_0}} - \frac{A_{j_0,k_0,l_{j_0,k_0}}}{(l_{j_0,k_0} - 1)!} \frac{q_{l_{j_0,k_0}-1}(\cos \pi \alpha_{j_0,k_0})}{\sin^{l_{j_0,k_0}}(\pi \alpha_{j_0,k_0})} \right) + \ldots = 0,$$

where $A_{j_0,k_0,l_{j_0,k_0}} \neq 0$ by (15). Now applying Lemma 3 we easily see that the term containing π to the maximal power doesn't vanish and we get a contradiction with the algebraic independence of π and $e^{\pi\sqrt{d}}$. This completes the proof. ☐

Remark 2.1. If under the assumptions of Theorem 2 we have $r_1 = \ldots = r_s = -1$, then T is either zero or transcendental.

Remark 2.2. We note that there are alternative proofs of formulas (16), (21) based on application of the residue theorem to the complex integrals

$$\frac{1}{2\pi i}\int_{L_N}\left(\sum_{j=1}^{s}\frac{P_j(z)}{Q_j(z)}\right)\Big(\pi\cot\pi z\Big)\,dz \quad\text{and}\quad \frac{1}{2\pi i}\int_{L_N}\left(\sum_{j=1}^{s}\frac{P_j(z)}{Q_j(z)}\right)\frac{\pi}{\sin\pi z}\,dz,$$

where L_N is a square contour with the vertices $(N+1/2)(\pm 1 \pm i)$. (See also [3, Theorem 2]).

Corollary 5. *Let* $a, b \in \mathbb{Z}$, $4b > a^2$, *and* $m \in \mathbb{N}$. *Then for any polynomial* $P(x) \in \overline{\mathbb{Q}}[x]$ *such that* $\deg P(x) < 2m$, $P(-x) = (-1)^a P(x-a)$, *the sum*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot P(n)}{(n^2+an+b)^m}$$

is transcendental. In particular, if $k \in \mathbb{Z}, 0 \le k < 2m$, *and the numbers* k, a *have the same parity, then the sum*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\left(n+\frac{a}{2}\right)^k}{(n^2+an+b)^m}$$

is transcendental.

Theorem 3. *Let* $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ *be periodic with period* $q \in \mathbb{N}$. *Suppose that* $r \in \mathbb{Z}$, $m, l_1, \ldots, l_m \in \mathbb{N}$, $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$,

$$\frac{P(-x)}{Q(-x)} = \pm\frac{P(x+qr)}{Q(x+qr)}, \tag{22}$$

$Q(x) = (x-\alpha_0)\cdot\prod_{k=1}^{2m}(x-\alpha_k)^{l_k}$, *where* $\alpha_0 = qr/2$, $\alpha_k = a_k + ib_k\sqrt{d} \in \mathbb{Q}(i\sqrt{d})\setminus\mathbb{N}$, $k = 1, \ldots, 2m$, *are distinct,* $\alpha_{m+k} = qr - \alpha_k$, $l_{m+k} = l_k$, $b_k \ge 0$, $k = 1, \ldots, m$, *and* f *is an even or odd function according to whether we have the sign "plus" or "minus" in* (22). *Suppose further that the series*

$$U = \sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}f(n)$$

converges. Then U *is either a computable algebraic number or transcendental. Moreover,* U *is transcendental if at least one of the following conditions holds:*

(i) $P(qr/2) = 0$ *and* $\displaystyle\sum_{\substack{t=1 \\ t-\alpha_k\notin q\mathbb{Z}}}^{q}\sum_{k=1}^{m} f(t)\cdot\operatorname*{res}_{z=\alpha_k}\frac{P(z)}{Q(z)} \ne 0,$

(ii) $P(qr/2) = 0$, $b_{k_0} := \min\{b_k > 0\}$ *is a unique minimum of the positive* *numbers* b_k, $\quad \operatorname*{res}_{z=\alpha_{k_0}} \frac{P(z)}{Q(z)} \neq 0 \quad$ *and* $\quad \sum\limits_{t=1}^{q} f(t) e^{\frac{-2\pi i t}{q}} \neq 0,$

$$(iii) \quad \sum_{\substack{t=1 \\ t-\alpha_k \notin q\mathbb{Z}}}^{q-1} \sum_{k=1}^{m} f(t) \cdot \operatorname*{res}_{z=\alpha_k} \frac{P(z)}{Q(z)} \neq \frac{i}{2} \frac{P(\frac{qr}{2})}{Q'(\frac{qr}{2})} \sum_{\substack{t=1 \\ t \neq q/2}}^{q-1} f(t) \cot\Big(\frac{\pi t}{q} + \pi\Big\{\frac{r}{2}\Big\}\Big) \quad \textit{and}$$

$P(qr/2) \neq 0$, $\quad$ *where* $\{x\}$ *denotes the fractional part of* $\ x$.

Proof. By Lemma 1, using the following partial fraction expansion:

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{2m} \sum_{l=1}^{l_k} \frac{A_{k,l}}{(x-\alpha_k)^l} + \frac{A_{0,1}}{x - \frac{qr}{2}},$$

where the coefficients $A_{k,l}$ are defined in (11) with α_k replaced by $-\alpha_k$ and $A_{0,1} = \frac{P(qr/2)}{Q'(qr/2)}$, we have

$$U = \sum_{t=1}^{q} f(t) \sum_{k=1}^{2m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!} \frac{A_{k,l}}{q^l} \psi^{(l-1)}\Big(\frac{t-\alpha_k}{q}\Big) - \frac{A_{0,1}}{q} \sum_{t=1}^{q} f(t) \psi\Big(\frac{t}{q} - \frac{r}{2}\Big).$$

By (22), for $1 \leq k \leq m$, $1 \leq l \leq l_k$, it easily follows that $A_{m+k,l} = \pm(-1)^l A_{k,l}$.

To prove the theorem, we first assume that $P(qr/2) = 0$. Then taking into account that $f(t) = \pm f(-t)$ and f is a q-periodic function we have

$$U = \sum_{t=1}^{q} f(t) \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!} \frac{A_{k,l}}{q^l} \Bigg(\psi^{(l-1)}\Big(\frac{t-\alpha_k}{q}\Big)$$

$$\pm (-1)^l \psi^{(l-1)}\Big(\frac{t-\alpha_{m+k}}{q}\Big)\Bigg) = \sum_{t=1}^{q} \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l f(t)}{(l-1)!} \frac{A_{k,l}}{q^l} \psi^{(l-1)}\Big(\frac{t-\alpha_k}{q}\Big)$$

$$+ \sum_{t=1}^{q} \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{A_{k,l} f(q-t)}{(l-1)! q^l} \psi^{(l-1)}\Big(\frac{t-\alpha_{m+k}}{q}\Big) = A + \sum_{t=1}^{q} f(t) \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-1)^l}{(l-1)!}$$

$$\times \frac{A_{k,l}}{q^l} \Bigg(\psi^{(l-1)}\Big(\frac{t-\alpha_k}{q}\Big) + (-1)^l \psi^{(l-1)}\Big(1 - r - \frac{t-\alpha_k}{q}\Big)\Bigg),$$

where $A = -f(q) \cdot \sum\limits_{k=1}^{m} \sum\limits_{l=1}^{l_k} \frac{A_{k,l}}{\alpha_{m+k}^l} \in \overline{\mathbb{Q}}$. Now by (3), (7) and Lemma 3 we get

$$U = C_0 + \sum_{\substack{t=1 \\ t-\alpha_k \in q\mathbb{Z}}}^{q} \sum_{k=1}^{m} \sum_{l=2}^{l_k} C_{t,k,l} \pi^l - \sum_{\substack{t=1 \\ t-\alpha_k \notin q\mathbb{Z}}}^{q} \sum_{k=1}^{m} \sum_{l=1}^{l_k} \frac{(-\pi)^l f(t) A_{k,l}}{q^l} p_{l-1} \Bigg(\cot\Big(\frac{\pi(t-\alpha_k)}{q}\Big)\Bigg)$$

with $C_0, C_{t,k,l} \in \overline{\mathbb{Q}}$, from which it follows that U is either equal to $C_0 \in \overline{\mathbb{Q}}$ or transcendental. If condition (i) or (ii) holds, then arguing as in the proof of Theorem 1 (i), (ii) we get that U is transcendental.

If $P(qr/2) \neq 0$, then $P(-x) = P(x + qr)$ and thus f is an odd function by the hypothesis. Arguing as above we deduce that $A_{k+m,l} = (-1)^{l-1} A_{k,l}$,

$1\le k\le m$, $1\le l\le l_k$, and

$$U=\sum_{t=1}^{q-1}f(t)\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{(-1)^l}{(l-1)!}\frac{A_{k,l}}{q^l}\Big(\psi^{(l-1)}\Big(\frac{t-\alpha_k}{q}\Big)+(-1)^l\psi^{(l-1)}\Big(1-r-\frac{t-\alpha_k}{q}\Big)\Big)-\frac{A_{0,1}}{2q}\sum_{t=1}^{q-1}f(t)\Big(\psi\Big(\frac{t}{q}-\frac{r}{2}\Big)-\psi\Big(1-\frac{t}{q}-\frac{r}{2}\Big)\Big).$$

As it is easily seen if q is even, then $f(q/2)=0$ and we may assume that $t\ne q/2$ in the last sum. Now by (2), for a positive integer $t\le q-1$, $t\ne q/2$, we have

$$\psi\Big(\frac{t}{q}-\frac{r}{2}\Big)=C+\psi\Big(\frac{t}{q}-\frac{r}{2}+\Big[\frac{r+1}{2}\Big]\Big),\quad \psi\Big(1-\frac{t}{q}-\frac{r}{2}\Big)=\tilde{C}+\psi\Big(1-\frac{t}{q}-\frac{r}{2}+\Big[\frac{r}{2}\Big]\Big),\tag{23}$$

where $C,\tilde{C}\in\overline{\mathbb{Q}}$ and $[x]$ denotes the integer part of x. Now by (3), (23) and Lemma 3 we get

$$U=C_1+\sum_{\substack{t=1\\ t-\alpha_k\in q\mathbb{Z}}}^{q-1}\sum_{k=1}^{m}\sum_{l=2}^{l_k}C_{t,k,l}\pi^l-\sum_{\substack{t=1\\ t-\alpha_k\notin q\mathbb{Z}}}^{q-1}\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}\pi^l f(t)}{(-q)^l}p_{l-1}\Big(\cot\Big(\frac{\pi(t-\alpha_k)}{q}\Big)\Big)+\frac{A_{0,1}\pi}{2q}\sum_{t=1}^{q-1}f(t)\cot\Big(\frac{\pi t}{q}+\pi\Big\{\frac{r}{2}\Big\}\Big)\tag{24}$$

with $C_1, C_{t,k,l}\in\overline{\mathbb{Q}}$ and therefore, U is either equal to C_1 or transcendental. If $r=0$, i.e., if $P(x)$ and $Q(x)$ are even and odd polynomials respectively, then $C_1=0$ and hence U is either zero or transcendental. If condition (iii) is valid, then the coefficient of π does not vanish in (24) and we conclude that U is transcendental. This completes the proof of the theorem. ☐

Remark 3.1. If under the assumptions of Theorem 3 we have $r=0$, then either $U=-f(q)\cdot\sum_{k=1}^{m}\sum_{l=1}^{l_k}\frac{A_{k,l}}{\alpha_{k+m}^l}$ or U is transcendental.

Theorem 4. *Let* $k\in\mathbb{N}$, $r\in\mathbb{Z}$, $qr/2\notin\mathbb{N}$, $P(x)\in\overline{\mathbb{Q}}[x]$ *and* $P(-x)=\pm P(x+qr)$. *Let* $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ *be an even or odd periodic function with period* $q\in\mathbb{N}$ *depending on whether* k *and* $\deg P(x)$ *have the same parity or not. Suppose further that the series*

$$U=\sum_{n=1}^{\infty}\frac{f(n)P(n)}{\big(n-\frac{qr}{2}\big)^k}$$

converges. Then the sum U *is either a computable algebraic number or transcendental. In particular, if* $r=0$, *then* U *is either zero or transcendental.*

Proof. For the rational function $P(x)/(x-qr/2)^k$ we have the following partial fraction expansion:

$$\frac{P(x)}{\left(x-\frac{qr}{2}\right)^k}=\sum_{l=0}^{[\frac{\deg P}{2}]}\frac{A_l}{\left(x-\frac{qr}{2}\right)^{k-\delta-2l}}\qquad\text{with}\quad A_l=\frac{1}{(2l+\delta)!}P^{(2l+\delta)}\left(\frac{qr}{2}\right)$$

and δ equal 0 or 1 according to whether $P(-x)=P(x+qr)$ or $P(-x)=-P(x+qr)$. Then by Lemma 1, we get

$$U=\sum_{t=1}^{q}f(t)\sum_{l=0}^{[\frac{\deg P}{2}]}\frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!}\frac{A_l}{q^{k-\delta-2l}}\psi^{(k-\delta-2l-1)}\left(\frac{t}{q}-\frac{r}{2}\right).$$

Note that if k and $\deg P$ have the same (distinct) parity, then $k-\delta$ is even (odd) and f is an even (odd) function by the hypothesis. Thus we have $f(t)=(-1)^{k-\delta}f(q-t)$ and

$$2U=\sum_{t=1}^{q}\sum_{l=0}^{[\frac{\deg P}{2}]}\frac{(-1)^{k-\delta-1}f(t)-f(q-t)}{(k-\delta-2l-1)!}\frac{A_l}{q^{k-\delta-2l}}\psi^{(k-\delta-2l-1)}\left(\frac{t}{q}-\frac{r}{2}\right)$$

or

$$\begin{aligned}2U=&\sum_{t=1}^{q-1}f(t)\sum_{l=0}^{[\frac{\deg P}{2}]}\frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!}\frac{A_l}{q^{k-\delta-2l}}\Bigg(\psi^{(k-\delta-2l-1)}\left(\frac{t}{q}-\frac{r}{2}\right)\\&+(-1)^{k-\delta}\,\psi^{(k-\delta-2l-1)}\left(1-\frac{t}{q}-\frac{r}{2}\right)\Bigg)+\tilde{U},\end{aligned}\tag{25}$$

where

$$\tilde{U}=\left(f(q)+(-1)^{k-\delta}f(0)\right)\sum_{l=0}^{[\frac{\deg P}{2}]}\frac{(-1)^{k-\delta-1}}{(k-\delta-2l-1)!}\frac{A_l}{q^{k-\delta-2l}}\psi^{(k-\delta-2l-1)}\left(1-\frac{r}{2}\right).\tag{26}$$

It can be easily seen that $\tilde{U}=0$ if f is an odd function; if f is even, then $k-\delta$ is even and by formulas (7) we have that

$$\tilde{U}=C+\sum_{l=0}^{[\frac{\deg P}{2}]}C_l\pi^{k-\delta-2l}\tag{27}$$

with algebraic coefficients C, C_l. From (23), (3), (7) and Lemma 3 it follows that

$$\psi^{(k-\delta-2l-1)}\left(\frac{t}{q}-\frac{r}{2}\right)+(-1)^{k-\delta}\psi^{(k-\delta-2l-1)}\left(1-\frac{t}{q}-\frac{r}{2}\right)\in\mathbb{Q}\,\pi^{k-\delta-2l}+\mathbb{Q}.\tag{28}$$

Finally, by (25)–(28), we find

$$U=\tilde{C}+\sum_{l=0}^{[\frac{\deg P}{2}]}\tilde{C}_l\pi^{k-\delta-2l}$$

with $\tilde{C}, \tilde{C}_l \in \overline{\mathbb{Q}}$ and therefore, either U is equal to $\tilde{C}$ or $U \notin \overline{\mathbb{Q}}$. If $r = 0$, then from (25), (26) it easily follows that U is either zero or transcendental. This completes the proof of the theorem. $\square$

The special case of Theorem 4 for the number $U = L(k, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k}$, where χ is an even (odd) Dirichlet character, was proved in [10, §6].

Now consider several applications of Theorem 3 which gives us means to construct new examples of transcendental numbers. If in Theorem 3 we put $f(n) = \chi(n)$, where $\chi(n)$ is a Dirichlet character $\mod q$, then the Gauss sum

$$\tau(\chi) = \sum_{k=1}^{q} \chi(k) e^{\frac{-2\pi i k}{q}}$$

is never zero when χ is a primitive character (see [4, Ch. 8]). Namely, we have that $|\tau(\chi)| = \sqrt{q}$. This gives us the following.

Corollary 6. *Let $q > 1$ be an integer and χ be a primitive character* $\mod q$. *Suppose that $P(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = \pm P(x + qr)$, $Q(x) = \prod_{k=1}^{2m} (x - \alpha_k)^{l_k}$ for some $m, l_1, \ldots, l_{2m} \in \mathbb{N}, r \in \mathbb{Z}$, where $\alpha_k = a_k + i b_k \sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{N}$, $k = 1, \ldots, 2m$, are distinct numbers such that $\alpha_{m+k} = qr - \alpha_k$, $b_k \geq 0$, $l_{m+k} = l_k$, $k = 1, 2, \ldots, m$, and χ is an even (odd) character if* $\deg P$ *is even (odd). If $b_{k_0} := \min\{b_k > 0\}$ is a unique minimum of the positive numbers b_k and $\operatorname*{res}_{z=\alpha_{k_0}} \frac{P(z)}{Q(z)} \neq 0$, then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \chi(n)$$

is transcendental.

Corollary 7. *Let $q > 1$ be a square-free integer, $q \equiv 1 \pmod 4$, and $\left(\frac{n}{q}\right)$ denote Jacobi's symbol. Then*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \left(\frac{n}{q}\right) \notin \overline{\mathbb{Q}},$$

where $P(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = P(x + qr)$ and $Q(x)$ is as in Corollary 6. *In particular, the sum*

$$\sum_{n=1}^{\infty} \frac{\left(\frac{n}{q}\right)}{(n^2 + qrn + b)^m}$$

is transcendental for any $m \in \mathbb{N}$, $b, r \in \mathbb{Z}$ such that $q^2 r^2 < 4b$.

Corollary 8. *Let $q > 1$ be a square-free integer, $q \equiv 3 \pmod 4$, and $\left(\frac{n}{q}\right)$ denote Jacobi's symbol. Then*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \left(\frac{n}{q}\right) \notin \overline{\mathbb{Q}},$$

where $P(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = -P(x+qr)$ *and* $Q(x)$ *is as in Corollary* 6*. In particular, the sum*

$$\sum_{n=1}^{\infty} \left(\frac{n}{q}\right) \frac{(n+\frac{qr}{2})^{2m-1}}{(n^2+qrn+b)^m}$$

is transcendental for any $m \in \mathbb{N}$, $b, r \in \mathbb{Z}$ *such that* $q^2r^2 < 4b$.

If χ_0 is the principal character $\mod q$, then

$$\sum_{n=1}^{q} \chi_0(n) = \varphi(q), \qquad \tau(\chi_0) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{-\frac{2\pi i k}{q}} = \mu(q),$$

where φ and μ are the Euler and Möbius functions, respectively (see [11, Ch. 3]) and we have

Corollary 9. *If* $q > 1$ *is a square-free integer and* χ_0 *is the principal character* $\mod q$, *then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \chi_0(n)$$

is transcendental, where $P(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = P(x+qr)$ *and the polynomial* $Q(x)$ *is as in Corollary* 6*. In particular, the sum of the series*

$$\sum_{n=1}^{\infty} \frac{\chi_0(n)}{(n^2+qrn+b)^m}$$

is transcendental for any $m \in \mathbb{N}$, $b, r \in \mathbb{Z}$ *such that* $q^2r^2 < 4b$.

Corollary 10. *Let* $q > 1$ *be an integer and* χ_0 *the principal character* $\mod q$. *Suppose that* $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$, $P(-x) = P(x+qr)$ *and* $Q(x) = \prod_{k=1}^{2m}(x-\alpha_k)^{l_k}$ *for some* $m, l_1, \ldots, l_{2m} \in \mathbb{N}, r \in \mathbb{Z}$, *where* $\alpha_k = a_k + ib_k\sqrt{d} \in \mathbb{Q}(i\sqrt{d}) \setminus \mathbb{Q}$, $k = 1, \ldots, 2m$, *are distinct and such that* $\alpha_{k+m} = \overline{\alpha_k}$, $b_k \geq 0$, $l_{k+m} = l_k$, $k = 1, 2, \ldots, m$. *If* $\sum_{k=1}^{m} \operatorname*{res}_{z=\alpha_k} \frac{P(z)}{Q(z)} \neq 0$, *then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} \chi_0(n)$$

is transcendental.

Corollary 11. *Let* $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ *be odd, periodic with period* $q \in \mathbb{N}$. *Then the sum*

$$\sum_{n=1}^{\infty} \frac{P(n)f(n)}{n(n^2+b)^m}$$

is either zero or transcendental for any $m, b \in \mathbb{N}$ *and any even polynomial* $P(x)$ *with* $\deg P \leq 2m$.

4. Transcendence of trigonometric series

Theorem 5. *Suppose that* $\beta_1,\dots,\beta_s\in[0,2)$ *are distinct rational numbers,* $Q(x),P_1(x),\ \dots,P_s(x)\in\overline{\mathbb{Q}}[x]$, $Q(x)=(x-\alpha_1)^{l_1}\dots(x-\alpha_m)^{l_m}$, *where* $\alpha_1,\dots,\alpha_m\in\mathbb{Q}(i\sqrt{d})\setminus\mathbb{Z}$ *are distinct,* $l_1,\dots,l_m\in\mathbb{N}$, $h(n)=\sum\limits_{j=1}^{s}P_j(n)e^{i\pi\beta_jn}$, *and for* $1\le j\le s$

$$\deg P_j(x)\le\begin{cases}\deg Q(x)-1 & \text{if}\quad \beta_j>0,\\ \deg Q(x)-2 & \text{if}\quad \beta_j=0,\ .\end{cases}$$

Then the sum

$$V=\sum_{n=-\infty}^{+\infty}\frac{h(n)}{Q(n)}$$

is either zero or transcendental.

Proof. We consider the complex integral

$$I_N=\frac{1}{2\pi i}\int_{L_N}\frac{h^-(z)}{Q(z)}\cdot\frac{\pi}{\sin\pi z}\,dz,$$

where $h^-(z)=\sum_{j=1}^{s}P_j(z)e^{i\pi(\beta_j-1)z}$, L_N is a square contour with the vertices $(N+1/2)(\pm1\pm i)$, and N is a large positive integer such that $\alpha_1,\dots,\alpha_m$ are inside L_N. For $z=\pm(N+1/2)+iy$, $y\in[-N-1/2,N+1/2]$ we have

$$\Bigl|\frac{1}{\sin\pi z}\Bigr|=\frac{2}{e^{\pi y}+e^{-\pi y}}$$

and therefore,

$$\Bigl|\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)\sin\pi z}\Bigr|=\frac{2|P_j(z)|}{|Q_j(z)|\Bigl(e^{\pi\beta_jy}+e^{\pi(\beta_j-2)y}\Bigr)}\le2\frac{|P_j(z)|}{|Q_j(z)|}e^{-\pi|y|\min\{\beta_j,2-\beta_j\}}. \tag{29}$$

If $\beta_j=0$, then from (29) it follows

$$\Bigl|\frac{1}{2\pi i}\int_{\substack{z=\pm(N+\frac12)+iy\\ -N-\frac12\le y\le N+\frac12}}\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)}\frac{\pi}{\sin\pi z}\,dz\Bigr|\le\int_{-N-\frac12}^{N+\frac12}\frac{|P_j(z)|}{|Q_j(z)|}\,dy=O\Bigl(\frac1N\Bigr). \tag{30}$$

If $0<\beta_j<2$, then (29) implies

$$\begin{aligned}\Bigl|\frac{1}{2\pi i}\int_{\substack{z=\pm(N+\frac12)+iy\\ -N-\frac12\le y\le N+\frac12}}\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)}\frac{\pi}{\sin\pi z}\,dz\Bigr|\le\\ \le O\Bigl(\frac1N\Bigr)\cdot\int_{-N-\frac12}^{N+\frac12}e^{-\pi|y|\min\{\beta_j,2-\beta_j\}}\,dy=O\Bigl(\frac1N\Bigr).\end{aligned} \tag{31}$$

If $z=x\pm i(N+1/2)$, $x\in[-N-1/2,N+1/2]$, then

$$\Bigl|\frac{1}{\sin\pi z}\Bigr|=\frac{2}{e^{\pi(N+\frac12)}-e^{-\pi(N+\frac12)}}$$

and

$$\left|\frac{P_j(z)e^{i\pi(\beta_j-1)z}}{Q(z)\sin\pi z}\right| \le \frac{2|P_j(z)|}{|Q(z)|}\frac{e^{\pi|\beta_j-1|(N+\frac{1}{2})}}{e^{\pi(N+\frac{1}{2})}-e^{-\pi(N+\frac{1}{2})}} \tag{32}$$
$$= \begin{cases} O\Big(\frac{1}{N^2}\Big), & \text{if } \beta_j = 0,\\ O\Big(\frac{1}{Ne^{\pi(1-|\beta_j-1|)N}}\Big), & \text{if } 0<\beta_j<2. \end{cases}$$

Therefore, by (30)–(32), we conclude that $I_N = O(N^{-1})$ as $N \to \infty$. On the other hand, by the residue theorem we have

$$I_N - \sum_{k=1}^{m} \operatorname*{res}_{z=\alpha_k}\left(\frac{h^-(z)}{Q(z)}\frac{\pi}{\sin\pi z}\right) = \sum_{k=-N}^{N} \operatorname*{res}_{z=k}\left(\frac{h^-(z)}{Q(z)}\frac{\pi}{\sin\pi z}\right) = \sum_{k=-N}^{N}\frac{h(k)}{Q(k)}.$$

Now letting N tend to infinity we get

$$V = -\sum_{k=1}^{m} \operatorname*{res}_{z=\alpha_k}\left(\frac{\pi\cdot h^-(z)}{Q(z)\sin\pi z}\right) = \sum_{k=1}^{m}\frac{-\pi}{(l_k-1)!}\left(\frac{h^-(z)\cdot(z-\alpha_k)^{l_k}}{Q(z)\sin\pi z}\right)^{(l_k-1)}\Big|_{z=\alpha_k},$$

which implies that $V \in \overline{\mathbb{Q}}(\pi, e^{\frac{\pi\sqrt{d}}{B}})$ for some $B\in\mathbb{N}$ and hence, either $V = 0$ or $V \notin \overline{\mathbb{Q}}$. □

Corollary 12. *If in addition to the assumptions of Theorem* 5, $Q(x)$ *is an even polynomial, then the sum*

$$W = \sum_{n=0}^{\infty}\frac{h(n)+h(-n)}{Q(n)}$$

is either $h(0)/Q(0)$ *or transcendental.*

Corollary 13. *Suppose that* $\beta_1, \beta_2 \in (0,1)\cup(1,2)$ *are rational numbers,* $Q(x)$, $P_1(x)$, $P_2(x) \in \overline{\mathbb{Q}}[x]$ *such that* $P_1(x), Q(x)$ *are even polynomials,* $P_2(x)$ *is an odd polynomial,* $\deg P_j(x) \le \deg Q(x) - 1$, $j = 1, 2$, *and all roots of the polynomial* $Q(x)$ *belong to* $\mathbb{Q}(i\sqrt{d})\setminus\mathbb{Z}$. *Then the trigonometric series*

$$W = \frac{P_1(0)}{2Q(0)} + \sum_{n=1}^{\infty}\frac{P_1(n)\cos(\pi\beta_1 n)+P_2(n)\sin(\pi\beta_2 n)}{Q(n)}$$

is either zero or transcendental.

Proof. We define

$$h(n) = \begin{cases} \frac{1}{2}P_1(n)e^{i\pi\beta_1 n} - \frac{i}{2}P_2(n)e^{i\pi\beta_2 n}, & \text{if} \quad \beta_1\ne\beta_2,\\ \frac{1}{2}P_1(n)e^{i\pi\beta_1 n} + \frac{i}{2}P_2(n)e^{i\pi(2-\beta_1)n}, & \text{if} \quad \beta_1=\beta_2 \end{cases}$$

and consider the sum

$$\sum_{n=0}^{\infty}\frac{h(n)+h(-n)}{Q(n)} - \frac{h(0)}{Q(0)} = \frac{P_1(0)}{2Q(0)} + \sum_{n=1}^{\infty}\frac{P_1(n)\cos(\pi\beta_1 n)+P_2(n)\sin(\pi\beta_2 n)}{Q(n)},$$

which, by Corollary 12 is either zero or transcendental. □

5. Schanuel's conjecture and infinite sums

For more general set of roots of the polynomials $Q_j(x)$, when not all $\alpha_{j,k}$ in $\mathbb{Q}(i\sqrt{d})$, we give some statements on the transcendence of the sums S, T, U, V provided that the Schanuel conjecture holds (see [12, §3.1], [8, §10.7.G]).

Conjecture (Schanuel) (S). *If* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$, *then the transcendence degree over* $\mathbb{Q}$ *of the field* $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ *is at least* n.

We formulate the following propositions, which are consequences of (S) :

Conjecture ($\mathbf{S_1}$). *Let* $P_1, \ldots, P_s,\ Q_1, \ldots, Q_s \in \overline{\mathbb{Q}}[x]$, $r_1, \ldots, r_s \in \mathbb{Z}$ *and for any* $1 \le j \le s$ *the polynomials* P_j, Q_j *satisfy the following conditions:* $\deg P_j \le \deg Q_j - 2$, $Q_j(r_j/2) \ne 0$, $Q_j(n) \ne 0$, $n = 0, 1, 2, \ldots,$ *and*

$$\frac{P_j(-x)}{Q_j(-x)} = \frac{P_j(r_j + x)}{Q_j(r_j + x)}.$$

Then the sum

$$S = \sum_{n=0}^{\infty}\Big(\frac{P_1(n)}{Q_1(n)} + \ldots + \frac{P_s(n)}{Q_s(n)}\Big)$$

is either a computable algebraic number or transcendental.

Proof. Under the conditions stated above, we see that for $1 \le j \le s$, $Q_j(x) = \prod_{k=1}^{2m_j}(x - \alpha_{j,k})^{l_{j,k}}$, where $\alpha_{j,k}$ are distinct algebraic numbers distinct from non-negative integers and such that $\alpha_{j,m_j+k} = r_j - \alpha_{j,k}$, $l_{j,m_j+k} = l_{j,k} \in \mathbb{N}$, $k = 1, 2, \ldots, m_j$. Therefore, from (16) we have

$$S = C_0 + \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\in\mathbb{Z}}}^{m_j}\sum_{\substack{l=1\\ l\, even}}^{l_{j,k}} C_{j,k,l}\pi^l + \sum_{j=1}^{s}\sum_{\substack{k=1\\ \alpha_{j,k}\notin\mathbb{Z}}}^{m_j}\sum_{l=1}^{l_{j,k}} \frac{(-1)^{l-1}A_{j,k,l}}{(l-1)!}\pi^l \cdot p_{l-1}(-\cot \pi\alpha_{j,k}), \tag{33}$$

where C_0 and all the coefficients $C_{j,k,l}, A_{j,k,l}$ are algebraic numbers. From (33) it follows that S is equal to C_0 or transcendental by (S). Indeed, suppose that $S \ne C_0$ and S is algebraic. Assume that the numbers

$$\frac{1}{\lambda}, \frac{\alpha_{j_1,k_1}}{\lambda_1}, \ldots, \frac{\alpha_{j_l,k_l}}{\lambda_l}, \tag{34}$$

where $\lambda_1, \ldots, \lambda_l \in \mathbb{N}$, are linearly independent over $\mathbb{Q}$ and all the other roots $\alpha_{j,k}$ are $\mathbb{Z}$-linear combinations of (34). Then the numbers

$$\frac{\pi i}{\lambda}, \frac{\pi i\alpha_{j_1,k_1}}{\lambda_1}, \ldots, \frac{\pi i\alpha_{j_l,k_l}}{\lambda_l}$$

are also linearly independent over $\mathbb{Q}$. Put

$$\begin{aligned} K &= \overline{\mathbb{Q}}\Big(\frac{\pi i}{\lambda}, \frac{\pi i\alpha_{j_1,k_1}}{\lambda_1}, \ldots, \frac{\pi i\alpha_{j_l,k_l}}{\lambda_l}, e^{\frac{\pi i\alpha_{j_1,k_1}}{\lambda_1}}, \ldots, e^{\frac{\pi i\alpha_{j_l,k_l}}{\lambda_l}}\Big) \\ &= \overline{\mathbb{Q}}\Big(\frac{\pi i}{\lambda}, e^{\frac{\pi i\alpha_{j_1,k_1}}{\lambda_1}}, \ldots, e^{\frac{\pi i\alpha_{j_l,k_l}}{\lambda_l}}\Big). \end{aligned}$$

Then by (S), it follows that tr $\deg(K:\overline{\mathbb{Q}})=l+1$. From (33) we have that $S-C_0\in K$. If $S-C_0\in\overline{\mathbb{Q}}\setminus\{0\}$, then there exists a non-zero polynomial $A(x)\in\mathbb{Z}[x]$ such that $A(S-C_0)=0$. Hence tr $\deg(K:\overline{\mathbb{Q}})\le l$ and the contradiction obtained proves (S_1). □

Remark 5.1. If all $\alpha_{j,k}\in\mathbb{Q}(i\sqrt{d})$, then (S_1) is true by Theorem 1.

By a similar argument we have

Conjecture (S₂). *Let* $P_1,\dots,P_s,\ Q_1,\dots,Q_s\in\overline{\mathbb{Q}}[x]$, $r_1,\dots,r_s\in\mathbb{Z}$ *and for any* $1\le j\le s$ *the polynomials* P_j,Q_j *satisfy the following conditions:* $\deg P_j\le\deg Q_j-1$, $Q_j(r_j/2)\ne 0$, $Q_j(n)\ne 0$, $n=0,1,2,\dots,$ *and*

$$\frac{P_j(-x)}{Q_j(-x)}=(-1)^{r_j}\frac{P_j(r_j+x)}{Q_j(r_j+x)}.$$

Then the sum

$$T=\sum_{n=0}^{\infty}\Big(\frac{P_1(n)}{Q_1(n)}+\ldots+\frac{P_s(n)}{Q_s(n)}\Big)(-1)^n$$

is either a computable algebraic number or transcendental.

Conjecture (S₃). *Let* $f:\mathbb{Z}\to\overline{\mathbb{Q}}$ *be periodic with period* $q\in\mathbb{N}$. *Suppose that* $r\in\mathbb{Z}$, $P(x),Q(x)\in\overline{\mathbb{Q}}[x]$, $(Q'(qr/2))^2+(Q(qr/2))^2\ne 0$, $Q(n)\ne 0$, $n=1,2,\dots,$

$$\frac{P(-x)}{Q(-x)}=\pm\frac{P(x+qr)}{Q(x+qr)} \tag{35}$$

and f *is an even or odd function according to whether we have the sign "plus" or "minus" in* (35). *Suppose further that the series*

$$U=\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}f(n)$$

converges. Then U *is either a computable algebraic number or transcendental.*

Conjecture (S₄). *Suppose that* $\beta_1,\dots,\beta_s\in[0,2)$ *are distinct rational numbers,* $Q(x),P_1(x),\dots,P_s(x)\in\overline{\mathbb{Q}}[x]$, $Q(n)\ne 0$, $n\in\mathbb{Z}$, $h(n)=\sum_{j=1}^{s}P_j(n)e^{i\pi\beta_j n}$, *and for* $1\le j\le s$, $\deg P_j(x)\le\deg Q(x)-1$ *if* $0<\beta_j<2$ *and* $\deg P_j(x)\le\deg Q(x)-2$ *if* $\beta_j=0$. *Then the sum*

$$V=\sum_{n=-\infty}^{+\infty}\frac{h(n)}{Q(n)}$$

is either zero or transcendental.

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